THE LOWER CENTRAL SERIES OF THE GROUP

$$\langle x, y: x^p = 1 \rangle$$

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ABSTRACT. In this paper we determine the Lie algebra associated to the lower central series of the group $\langle x, y: x^p = 1 \rangle$, p a prime.

1. Statement of results. Let F be the free group on x_1, x_2 and let $G = F/R$, where $R = (x_1^p)$ is the normal subgroup of F generated by x_1^p , p a prime. If $\mathfrak{L}, \mathfrak{g}$ are, respectively, the graded Lie algebras associated to the lower central series of F, G (cf. [1], [7], [9]), then $\mathfrak{L}$ is the free Lie algebra (over $\mathbb{Z}$) on ξ_1, ξ_2 , the images of x_1, x_2 , respectively, in $\mathfrak{L}_1 (= F/[F, F])$. If U is the enveloping algebra of $\mathfrak{L}$, we view $\mathfrak{L}$ as a left U -module via the adjoint representation $\text{ad}: U \rightarrow \text{End}_{\mathbb{Z}}(\mathfrak{L})$ and identify $\mathfrak{L}$ with its image in U under the canonical injection $\mathfrak{L} \rightarrow U$.

THEOREM. *If $\mathfrak{r}$ is the kernel of the canonical surjection $\mathfrak{L} \rightarrow \mathfrak{g}$, then $\mathfrak{r}$ is generated, as an ideal of $\mathfrak{L}$, by the elements*

$$p\xi_1, \quad [\text{ad}(\lambda\xi_1)\xi_2, \text{ad}(\lambda\xi_1^{n(p-1)+p})\xi_2] \quad (\lambda \in U, n \geq 0).$$

This theorem answers a special case of a question posed by Magnus concerning power relations (cf. [9, p. 374]). As a by-product of the proof, we determine the dimensions of the homogeneous components $\mathfrak{g}_n$ for $n \geq 2$ as vector spaces over the field $\mathbb{F}_p$ of p -elements. This was done in [2] by entirely different methods.

2. Preliminaries. Let M be the Magnus algebra on $X_1, \dots, X_n$, i.e., the algebra of noncommutative formal power series in $X_1, \dots, X_n$ with coefficients in $\mathbb{Z}$. Let $(t_1, \dots, t_n)$ be an n -tuple of positive real numbers $t_i > 0$, and define a function

$$w: M \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$$

by

$$w\left(\sum a_{i_1 \dots i_k} X_{i_1} \cdots X_{i_k}\right) = \inf\{t_{i_1} + \cdots + t_{i_k} + v_p(a_{i_1 \dots i_k}): a_{i_1 \dots i_k} \neq 0\},$$

where v_p is the p -adic valuation on $\mathbb{Z}$ ($v_p(p) = 1$). For each real number

Received by the editors June 22, 1976.

AMS (MOS) subject classifications (1970). Primary 20F35, 20F40; Secondary 17B35, 17B65.

Key words and phrases. One relator group, lower central series, Lie algebra.

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$\alpha \geq 0$ let $I_\alpha = \{u \in M: w(u) \geq \alpha\}$. Then $I_\alpha \subseteq I_\beta$ if $\alpha \geq \beta$, and $I_\alpha I_\beta \subseteq I_{\alpha+\beta}$ which implies that I_α is an ideal of M . Since $p \in I_1$, we have $pI_\alpha \subseteq I_{\alpha+1}$. Let $I_\alpha^+ = \bigcup_{\beta > \alpha} I_\beta$, let $\text{gr}_\alpha(M) = I_\alpha / I_\alpha^+$, and let

$$\text{gr}(M) = \bigoplus_{\alpha > 0} \text{gr}_\alpha(M).$$

Then $\text{gr}(M)$ is a graded algebra over $\mathbb{F}_p$ and π , the image of p in $\text{gr}_1(M)$, is transcendental over $\mathbb{F}_p$. Moreover, if ξ_i is the image of X_i in $\text{gr}_1(M)$, then $\text{gr}(M)$ is the free associative algebra over $\mathbb{F}_p[\pi]$ on $\xi_1, \dots, \xi_n$.

If F is the free group on $x_1, \dots, x_n$, then the mapping of F into the group of units of M which sends x_i to $1 + X_i$ is injective (cf. [1], [7], [9]), and we use this injection to identify F with a subgroup of the group of units of M . For any real number $\alpha > 0$, let $F_\alpha = (1 + I_\alpha) \cap F$. Then the F_α are subgroups of F with

- (i) $\bigcup_{\alpha > 0} F_\alpha = F$,
- (ii) $F_\alpha \subseteq F_\beta$ if $\alpha \geq \beta$,
- (iii) $[F_\alpha, F_\beta] \subseteq F_{\alpha+\beta}$, where $[H, K]$ is the subgroup of F generated by the commutators $[x, y] = x^{-1}y^{-1}xy$ with $x \in H, y \in K$. Let

$$F_\alpha^+ = \bigcup_{\beta > \alpha} F_\beta.$$

Then F_α^+ is a normal subgroup of F with F_α / F_α^+ abelian; in fact, $[F_\alpha, F_\beta] \subseteq F_\alpha^+$ for any $\beta > 0$. Let $\text{gr}_\alpha(F)$ be the abelian group F_α / F_α^+ , written additively, and let $j_\alpha: F_\alpha \rightarrow \text{gr}_\alpha(F)$ be the canonical surjection. If $\xi = j_\alpha(x)$, $\eta = j_\beta(y)$, then

$$[\xi, \eta] = j_{\alpha+\beta}([x, y])$$

uniquely defines an element $[\xi, \eta] \in \text{gr}_{\alpha+\beta}(F)$, and this bracket operation yields a Lie algebra structure (over $\mathbb{Z}$) on the graded abelian group $\text{gr}(F) = \bigoplus_{\alpha > 0} \text{gr}_\alpha(F)$, (cf. [1], [8]). Since $x^p \in F_\alpha^+$ if $x \in F_\alpha$, we have that $\text{gr}(F)$ is a Lie algebra over $\mathbb{F}_p$. Now the inclusion $F \subseteq M$ induces a mapping

$$\theta: \text{gr}(F) \rightarrow \text{gr}(M), \quad x \pmod{F_\alpha^+} \rightarrow x - 1 \pmod{I_\alpha^+}$$

which is an injective Lie algebra homomorphism (over $\mathbb{F}_p$) of $\text{gr}(F)$ into the underlying Lie algebra of $\text{gr}(M)$, i.e.,

$$\theta([\xi, \eta]) = \theta(\xi)\theta(\eta) - \theta(\eta)\theta(\xi) = [\theta(\xi), \theta(\eta)].$$

We use this map to identify $\text{gr}(F)$ with its image in $\text{gr}(M)$.

For $\alpha > 0$ let $\varphi(\alpha) = \min(p\alpha, \alpha + 1)$ and let $P_\alpha: \text{gr}_\alpha(M) \rightarrow \text{gr}_{\varphi(\alpha)}(M)$ be the map defined by

$$P_\alpha(\eta) = \begin{cases} \eta^p & \text{if } \alpha < 1/(p-1), \\ \eta^p + \pi\eta & \text{if } \alpha = 1/(p-1), \\ \pi\eta & \text{if } \alpha > 1/(p-1). \end{cases}$$

If $\eta = j_\alpha(y) \in \text{gr}_\alpha(F)$, then $P_\alpha(\eta) = y^p \pmod{F_{\alpha+1}^+}$. The Lie algebra $\text{gr}(F)$ is the smallest graded $\mathbb{F}_p$ -Lie subalgebra of $\text{gr}(M)$ stable under the P_α and

containing $\bar{\xi}_1, \dots, \bar{\xi}_n$ (cf. [8]). If $t_1 = \dots = t_n = 1$, then $(F_n)_{n \geq 1}$ is the lower p -central series of F , i.e. $F_1 = F$, $F_{n+1} = F_n^p[F, F_n]$ (loc. cit.).

3. Proof of theorem. In the notation of §2, we take $n = 2$, $t = t_1 = t_2 = 1/(p-1)$. We let $C_n F$ denote the n th term of the lower central series of F ; we have $C_1 F = F$, $C_{n+1} F = [F, C_n F]$. The inclusion $C_n F \subseteq F_n$ defines a Lie algebra homomorphism (over $\mathbb{Z}$) $\psi: \mathfrak{L} \rightarrow \text{gr}(F)$ whose kernel is $p\mathfrak{L}$ and whose image is the free Lie algebra $\bar{\mathfrak{L}}$ on $\bar{\xi}_1, \bar{\xi}_2$ over $\mathbb{F}_p$.

Let S be the normal subgroup of F generated by $s = [x_1^p, x_2]$. If

$$\mathfrak{s} = \bigoplus_{n \geq 1} \mathfrak{s}_n,$$

where $\mathfrak{s}_n$ is the image of $S \cap C_n F$ in $\mathfrak{L}_n$, we have $\mathfrak{s}_1 = 0$ and $\mathfrak{s}_n = \mathfrak{r}_n$ for $n \geq 2$. We therefore have $\mathfrak{r} = \mathfrak{r}_1 \oplus \mathfrak{s}$ with $\mathfrak{r}_1 \cong \mathbb{Z}$ generated by $p\xi_1$. Now let

$$\mathfrak{s}' = \bigoplus_{\alpha > 0} \mathfrak{s}'_{\alpha},$$

where $\mathfrak{s}'_{\alpha}$ is the image of $S \cap F_{\alpha}$ in $\text{gr}_{\alpha}(F)$. The image of $s = [x_1^p, x_2]$ in $\text{gr}_{3t}(F)$ is

$$\sigma = \pi[\bar{\xi}_1, \bar{\xi}_2] + \text{ad}(\bar{\xi}_1)^p \bar{\xi}_2;$$

in fact, by an identity of P. Hall (cf. [3]),

$$[x_1^p, x_2] = [x_1, x_2]^p c_3^p \cdots c_p^p c_{p+1}$$

with $c_i \in C_i F$ and $\text{ad}(\bar{\xi}_1)^p \bar{\xi}_2$ equal to the image of c_{p+1} in $\text{gr}_{(p+1)t}(F)$.

LEMMA 1. *Given $k \geq 1$, $m \geq 2$, $v \in C_m F$, there exists $u \in S \cap F_{k+mt} \cap C_m F$ such that*

$$u \equiv v^{p^k} \pmod{C_{m+1} F}.$$

PROOF. If $z_1, \dots, z_n \in F$ and $\text{ad}(x)y = [x, y]$, then

$$(\text{ad}(z_1) \cdots \text{ad}(z_n)s)^{p^{k-1}} \equiv (\text{ad}(z_1) \cdots \text{ad}(z_n)\text{ad}(x_1)x_2)^{p^k} \pmod{C_{n+3} F}$$

and $(\text{ad}(z_1) \cdots \text{ad}(z_n)s)^{p^{k-1}} \in S \cap F_{k+(n+2)t} \cap C_{n+2} F$, from which the lemma immediately follows.

LEMMA 2. $\psi(\mathfrak{s}) = \mathfrak{s}' \cap \bar{\mathfrak{L}}$.

PROOF. Since $S \cap C_n F \subseteq S \cap F_n$ we have $\psi(\mathfrak{s}) \subseteq \mathfrak{s}' \cap \bar{\mathfrak{L}}$. Now let $\zeta \in (\mathfrak{s}' \cap \bar{\mathfrak{L}})_{\alpha}$, $\zeta \neq 0$, and let $z \in S \cap F_{\alpha}$ with image in $\text{gr}_{\alpha}(F)$ equal to ζ . Assume z chosen so that the integer m with $z \in C_m F$, $z \notin C_{m+1} F$ is as large as possible. If $\alpha = nt$, we have $m \leq n$. If $m < n$, then

$$z \equiv \prod_{i=1}^q v_i^{p^{k_i}} \pmod{C_{m+1} F}$$

with $v_i \in C_m F$ and $k_i + mt > \alpha$. By Lemma 1, there is an element $u \in S \cap F_{\alpha}^+ \cap C_m F$ such that

$$u \equiv \prod_{i=1}^q v_i^{p^{k_i}} \pmod{C_{m+1} F}.$$

Then $z' = u^{-1}z \in C_{m+1} F$ and $z' \equiv z \pmod{F_\alpha^+}$. Hence we must have had $m = n$. If ρ is the image of z in $\mathcal{C}_n = C_n F / C_{n+1} F$, then $\rho \in \mathfrak{s}_n$ and $\psi(\rho) = \zeta$.

Let $\mathfrak{S}'$ be the $\mathbb{F}_p[\pi]$ -Lie subalgebra of $\text{gr}(M)$ generated by ξ_1, ξ_2 . Then (cf. [4, Theorem 4']) $\mathfrak{s}'$ is the ideal of $\mathfrak{S}'$ generated by σ . Let $\bar{U}$ be the enveloping algebra of $\bar{\mathfrak{S}}$, i.e. the $\mathbb{F}_p$ -subalgebra of $\text{gr}(M)$ generated by $\bar{\xi}_1, \bar{\xi}_2$. Since $p\text{gr}_n(F) \subseteq \mathfrak{s}_n$ for $n \geq 2$, the theorem is an immediate consequence of the following lemma.

LEMMA 3. $\bar{\mathfrak{S}} \cap \mathfrak{s}'$ is the ideal of $\bar{\mathfrak{S}}$ generated by the elements

$$\left[\text{ad}(\lambda \bar{\xi}_1) \bar{\xi}_2, \text{ad}(\lambda \bar{\xi}_1^{n(p-1)+p}) \bar{\xi}_2 \right] \quad (\lambda \in \bar{U}, n \geq 0).$$

PROOF. If $\lambda \in \text{gr}(M)$, there are $\lambda_0, \lambda_1, \dots, \lambda_n \in \bar{U}$ such that $\lambda = \lambda_0 + \pi \lambda_1 + \dots + \pi^n \lambda_n$. Then, if $\text{ad}: \text{gr}(M) \rightarrow \text{End}_{\mathbb{F}_p[\pi]}(\mathfrak{S}')$ is the adjoint representation of $\mathfrak{S}'$, we have

$$\begin{aligned} \text{ad}(\lambda)\sigma &= \text{ad}(\lambda_0)\text{ad}(\bar{\xi}_1)^p \bar{\xi}_2 \\ &+ \sum_{i=1}^n \pi^i \text{ad}(\lambda_{i-1} + \lambda_i \bar{\xi}_1^{p-1}) [\bar{\xi}_1, \bar{\xi}_2] + \pi^{n+1} \text{ad}(\lambda_n) [\bar{\xi}_1, \bar{\xi}_2]. \end{aligned}$$

Hence $\text{ad}(\lambda)\sigma \in \bar{\mathfrak{S}}$ iff $\lambda_{i-1} + \lambda_i \bar{\xi}_1^{p-1}$ ($1 \leq i \leq n$) and λ_n annihilate $[\bar{\xi}_1, \bar{\xi}_2]$, i.e.

$$\text{ad}(\lambda_{i-1} + \lambda_i \bar{\xi}_1^{p-1}) [\bar{\xi}_1, \bar{\xi}_2] = \text{ad}(\lambda_n) [\bar{\xi}_1, \bar{\xi}_2] = 0.$$

But this holds iff for $0 \leq i \leq n$,

$$\lambda_{n-i} = \sum_{j=0}^i (-1)^j \rho_{i-j} \bar{\xi}_1^{j(p-1)}$$

with $\text{ad}(\rho_j)[\bar{\xi}_1, \bar{\xi}_2] = 0$. It follows that the elements $\text{ad}(\rho \bar{\xi}_1^{n(p-1)}) \text{ad}(\bar{\xi}_1^p) \bar{\xi}_2$ with $n \geq 0, \rho \in \bar{U}, \text{ad}(\rho)[\bar{\xi}_1, \bar{\xi}_2] = 0$ generate $\bar{\mathfrak{S}} \cap \mathfrak{s}'$ as a vector space over $\mathbb{F}_p$ since $\mathfrak{s}' = \{\text{ad}(\lambda)\sigma: \lambda \in \text{gr}(M)\}$. Now a result of [6] states that the annihilator of $[\bar{\xi}_1, \bar{\xi}_2]$ in $\bar{U}$ is the left ideal of $\bar{U}$ generated by the elements $(\text{ad}(\lambda)[\bar{\xi}_1, \bar{\xi}_2])\lambda$ ($\lambda \in \bar{U}$). Hence $\bar{\mathfrak{S}} \cap \mathfrak{s}'$ is the ideal of $\bar{\mathfrak{S}}$ generated by the elements

$$\begin{aligned} &\text{ad}\left((\text{ad}(\lambda)[\bar{\xi}_1, \bar{\xi}_2])\lambda \bar{\xi}_1^{n(p-1)}\right) \text{ad}(\bar{\xi}_1^p) \bar{\xi}_2 \\ &= \left[\text{ad}(\lambda \bar{\xi}_1) \bar{\xi}_2, \text{ad}(\lambda \bar{\xi}_1^{n(p-1)+p}) \bar{\xi}_2 \right] \end{aligned}$$

with $n \geq 0, \lambda \in \bar{U}$. Q.E.D.

Since $\mathfrak{r}_n \supseteq p\text{gr}_n(F)$ for $n \geq 2$ we see that $\mathfrak{g}_n$ is a vector space over $\mathbb{F}_p$ for $n \geq 2$. To obtain a formula for its dimension we use the fact that for $\alpha \geq (p+1)t$,

$$\mathfrak{s}'_\alpha + \bar{\mathfrak{S}}_\alpha = \mathfrak{S}'_\alpha.$$

This yields

$$\dim(\mathfrak{s}'_\alpha \cap \bar{\mathfrak{L}}_\alpha) = \dim \mathfrak{s}'_\alpha + \dim \bar{\mathfrak{L}}_\alpha - \dim \mathfrak{L}'_\alpha,$$

and hence that

$$\dim \mathfrak{g}_n = \dim(\mathfrak{L}'_{nt}/\mathfrak{s}'_{nt})$$

for $n \geq p + 1$, as we see using the fact that

$$\psi(\mathfrak{s}_n) = \bar{\mathfrak{L}}_{nt} \cap \mathfrak{s}'_{nt}.$$

Using results of [4] we have

$$\dim \mathfrak{L}'/\mathfrak{s}' = \dim \mathfrak{L}'/\mathfrak{t}',$$

where $\mathfrak{t}'$ is the ideal of $\mathfrak{L}'$ generated by $\tau = \text{ad}(\bar{\xi}_1)^p \bar{\xi}_2$. But

$$\mathfrak{L}'/\mathfrak{t}' \cong (\bar{\mathfrak{L}}/\bar{\mathfrak{t}}) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\pi].$$

where $\bar{\mathfrak{t}}$ is the ideal of $\bar{\mathfrak{L}}$ generated by τ . Hence, if $h_n = \dim \bar{\mathfrak{L}}_{nt}/\bar{\mathfrak{t}}_{nt}$, we have for $n \geq p + 1$,

$$\dim \mathfrak{g}_n = \sum_{k=1}^n h_n,$$

where (cf. [5])

$$nh_n = \sum_{d|n} \mu(n/d) \left(\sum_{0 \leq i \leq [d/(p+1)]} (-1)^i \frac{d}{d-pi} \binom{d-pi}{i} 2^{d-(p+1)i} \right).$$

By inspection, the above formula for $\dim \mathfrak{g}_n$ also holds for $2 \leq n \leq p$.

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