INVERSE LIMITS AND ABSOLUTES OF H -CLOSED SPACES

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ABSTRACT. The inverse limit of an inverse system of nonempty (respectively, minimal Hausdorff) H -closed spaces with continuous open surjective bonding maps is shown to be a nonempty H -closed (respectively, minimal Hausdorff) space. This result is used to resolve a conjecture, in the affirmative, of S. W. Willard: Every H -closed space is the continuous and open image of a minimal Hausdorff space.

1. Introduction. A space (all spaces in this paper are assumed to be Hausdorff spaces) is H -closed (HC) or *absolutely closed* if it is closed in any Hausdorff space in which it is embedded. A space is *minimal* Hausdorff (MH) if no proper subtopology is Hausdorff. The continuous image of a MH space is known to be HC; S. W. Willard [W, p. 270] has conjectured that, conversely, every HC space is the continuous image of some MH space. In this paper we answer Willard's conjecture affirmatively by constructing the required MH space using inverse limit techniques. We also show that the inverse limit of an inverse system of nonempty HC (MH) spaces with open continuous surjective bonding maps is nonempty and HC (MH). This result is the principal tool employed to answer Willard's conjecture.

Our construction was motivated by the observation that for several well-known examples of HC spaces X , there is an open set U of X such that if disjoint copies of X are identified on the complement of U , the resultant quotient space, (X, U) , is a MH space which maps onto X by a continuous open map. See Examples (4.1) and (4.2). Unfortunately, we have not been able to construct such an open set U for an arbitrary HC space X . Under these circumstances, we naturally turn to inverse limits in order to "regularize" each of the open sets of X , one at a time.

2. Inverse limits of HC spaces. In this section we show that an inverse system of nonempty HC spaces with open continuous surjective bonding maps has a nonempty HC inverse limit. The result is obtained by showing that every such system induces an inverse system of nonempty compact spaces with continuous bonding maps with a nonempty compact inverse limit. We also show that the inverse limit of a system of nonempty MH spaces with open continuous surjective bonding maps is nonempty and MH.

(2.1) **NOTATION AND DEFINITIONS.** An *open filterbase* on a space X is a collection $\mathcal{U}$ of nonempty open subsets of X such that if $U_1, U_2 \in \mathcal{U}$, then

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$U_1 \cap U_2$ contains an element of $\mathcal{U}$. An *open filter* is an open filterbase $\mathcal{U}$ such that $U_1, U_2 \in \mathcal{U}$ imply $U_1 \cap U_2 \in \mathcal{U}$, and $U \in \mathcal{U}$, $U \subset W$, W open in X imply $W \in \mathcal{U}$. An *open ultrafilter* is an open filter properly contained in no open filter. Denote by $\mathcal{N}(p)$ the open filter of neighborhoods of p (when the containing space is understood). If $\mathcal{U}$ is an open filterbase on X and $p \in X$, we say $\mathcal{U}$ *converges* to p and write $\mathcal{U} \rightarrow p$ if each $N \in \mathcal{N}(p)$ contains some $U \in \mathcal{U}$. A function $f: X \rightarrow Y$ is *θ -continuous* if for every $x \in X$ and open set V containing $f(x)$, there is an open set U containing x such that $f(\overline{U}) \subset \overline{V}$.

The space θX of all open ultrafilters on X is topologized as follows: For U open in X let O_U denote the set of all open ultrafilters on X containing U ; the sets O_U form a basis for θX . It is known that θX is a compact, extremally disconnected Hausdorff space [IF, p. 40]. The space θX is sometimes referred to as the *hyperabsolute* of X ; the subspace ωX of θX consisting of the convergent elements is called the *absolute* of X . (Thus, since a space is HC if and only if every open ultrafilter converges, X is HC if and only if $\theta X = \omega X$.) Iliadis [I, pp. 295–296] has shown that ωX maps irreducibly, θ -continuously and perfectly onto X .

Let $\mathcal{Q}$ be a set directed by $<$. Suppose for each $\alpha \in \mathcal{Q}$ that X_α is a space and that for each pair $\alpha, \beta \in \mathcal{Q}$ with $\beta \leq \alpha$, there is a continuous function $\Phi_{\alpha\beta}: X_\alpha \rightarrow X_\beta$ such that $\Phi_{\alpha\alpha}$ is the identity map on X_α and $\gamma \leq \beta \leq \alpha$ implies $\Phi_{\alpha\gamma} = \Phi_{\beta\gamma} \Phi_{\alpha\beta}$. Then we say $\{X_\alpha, \Phi_{\alpha\beta}, \mathcal{Q}\}$ is an *inverse system*. A point $(p_\alpha) \in \prod_{\alpha \in \mathcal{Q}} X_\alpha$ is called a *thread* of the system if $\beta < \delta$ implies $p_\beta = \Phi_{\delta\beta}(p_\delta)$. The set of all threads is called the *inverse limit* of the system and is denoted by $\text{inv lim}\{X_\alpha, \Phi_{\alpha\beta}, \mathcal{Q}\}$ or simply by X_∞ . If $B \subset X_\beta$ for some $\beta \in \mathcal{Q}$, denote by $\langle B \rangle_\infty$ the set $\{(p_\alpha) \in X_\infty | p_\beta \in B\}$.

(2.2) If X is HC, then the natural map $e: \theta X \rightarrow X$ sending an open ultrafilter to the point to which it converges is a θ -continuous perfect surjection [I, p. 296].

(2.3) Let $\{X_\alpha, \Phi_{\alpha\beta}, \mathcal{Q}\}$ be an inverse system. Sets of the form $\langle V \rangle_\infty$, where V is open in some X_β , constitute a basis for X_∞ [C, p. 234].

(2.4) If $\{X_\alpha, \Phi_{\alpha\beta}, \mathcal{Q}\}$ is an inverse system of nonempty compact spaces and continuous functions, then its inverse limit is nonempty and compact [C, p. 235].

(2.5) Let $f: X \rightarrow Y$ be a continuous open surjection of HC spaces. Let $p \in X$, $q = f(p)$, and let $\mathcal{U}'$ be an open ultrafilter on Y converging to q . Then there exists $\mathcal{U} \in \theta X$ with $\mathcal{U} \rightarrow p$ and $f(\mathcal{U}) = \mathcal{U}'$ (that is, $\mathcal{U}' = \{f(U) | U \in \mathcal{U}\}$) [M, p. 290].

(2.6) Let $f: X \rightarrow Y$ be a continuous open surjection of HC spaces. Then there exists a unique continuous open surjection $f^*: \theta X \rightarrow \theta Y$ such that the diagram below commutes:

$$\begin{array}{ccc} \theta X & \xrightarrow{f^*} & \theta Y \\ \downarrow e & & \downarrow e \\ X & \xrightarrow{f} & Y \end{array}$$

PROOF. For any open ultrafilter $\mathcal{U}$ on X , define $f^*(\mathcal{U}) = \mathcal{U}' = \{f(U) | U \in \mathcal{U}\}$

$\in \mathcal{U}\}$. Then since f is open and continuous, $f^*(\mathcal{U})$ is an open ultrafilter on Y for any $\mathcal{U} \in \theta X$. To see that f^* is continuous, let $O_U = \{\mathcal{U}' \mid U \in \mathcal{U}'\}$ be a basic open set in θY . One sees that $f^{*-1}(O_U) = O_{f^{-1}(U)}$ which is open in θX . Also, f^* is open, since $f^*(O_U) = O_{f(U)}$. By (2.5), f^* is surjective. Since $\mathcal{U} \rightarrow p$ implies $f(\mathcal{U}) \rightarrow f(p)$, $fe = ef^*$ and the diagram commutes. Finally, since $f^*(O_U) = O_{f(U)}$, we see that f^* is the unique map satisfying the required conditions.

(2.7) *If $\{X_\alpha, \Phi_{\alpha\beta}, \mathcal{Q}\}$ is an inverse system with open maps $\Phi_{\alpha\beta}$, and if U is open in X_β , $\beta \in \mathcal{Q}$, then $\overline{\langle U \rangle_\infty} = \langle \bar{U} \rangle_\infty$.*

PROOF. Suppose $(p_\alpha) \notin \overline{\langle U \rangle_\infty}$. Then there is some open set V in some X_γ with $(p_\alpha) \in \langle V \rangle_\infty$ and $\langle U \rangle_\infty \cap \langle V \rangle_\infty = \emptyset$. Let $\delta > \beta$ and $\delta > \gamma$, and let $W_1 = \Phi_{\delta\beta}^{-1}(U)$ and $W_2 = \Phi_{\delta\gamma}^{-1}(V)$. Observe that $\langle U \rangle_\infty = \langle W_1 \rangle_\infty$ and $\langle V \rangle_\infty = \langle W_2 \rangle_\infty$, so $W_1 \cap W_2 = \emptyset$. Since $p_\delta \in W_2$, $p_\beta \in \Phi_{\delta\beta}(W_2)$, which is an open set missing U . Thus $(p_\alpha) \notin \overline{\langle U \rangle_\infty}$.

Conversely, if $(p_\alpha) \notin \overline{\langle U \rangle_\infty}$, then $p_\beta \notin \bar{U}$, so $(p_\alpha) \in \langle X_\beta - \bar{U} \rangle_\infty$, an open set missing $\langle U \rangle_\infty$. Thus $(p_\alpha) \notin \overline{\langle U \rangle_\infty}$.

(2.8) THEOREM. *The inverse limit of an inverse system of nonempty HC spaces with continuous open surjective bonding maps is a nonempty HC space.*

PROOF. Let $\{X_\alpha, \Phi_{\alpha\beta}, \mathcal{Q}\}$ be an inverse system of nonempty HC spaces and open surjections. For each pair $\alpha, \beta \in \mathcal{Q}$, $\beta < \alpha$, let $\Phi_{\alpha\beta}^*$ be the unique continuous open surjection of θX_α onto θX_β given by (2.6). Note that since $\Phi_{\alpha\beta}^*$ is the unique map satisfying (2.6), $\{\theta X_\alpha, \Phi_{\alpha\beta}^*, \mathcal{Q}\}$ is an inverse system, i.e. if $\delta < \beta < \alpha$, then $\Phi_{\beta\delta}^* \Phi_{\alpha\beta}^* = \Phi_{\alpha\delta}^*$. Hence, by (2.4), the inverse limit X^* of $\{\theta X_\alpha, \Phi_{\alpha\beta}^*, \mathcal{Q}\}$ is a nonempty compact Hausdorff space. We will show that the map $e_\infty: X^* \rightarrow X_\infty$, which assigns to each thread of ultrafilters the thread of their limits, is surjective and θ -continuous. To this end, let $\beta \in \mathcal{Q}$ and let V be open in X_β . We will show $e_\infty(\overline{\langle O_V \rangle_\infty}) \subset \overline{\langle V \rangle_\infty}$, which equals $\langle \bar{V} \rangle_\infty$ by (2.7). Suppose $e_\infty((\mathcal{U}_\alpha)) \notin \overline{\langle V \rangle_\infty}$. Then no point of $\bar{V}$ is an adherent point of $\mathcal{U}_\beta$, which implies $X_\beta - \bar{V} \equiv W \in \mathcal{U}_\beta$. So $\mathcal{U}_\beta \in O_W$ and $(\mathcal{U}_\alpha) \in \langle O_W \rangle_\infty$. But $\langle O_W \rangle_\infty \cap \langle O_V \rangle_\infty = \emptyset$, so $(\mathcal{U}_\alpha) \notin \langle O_V \rangle_\infty$, and e_∞ is θ -continuous.

Now let $(p_\alpha) \in X_\infty$. Let $e_\alpha: \theta X_\alpha \rightarrow X_\alpha$ be the θ -continuous evaluation function of (2.2). By (3.4) of [DP], $P_\alpha \equiv e_\alpha^{-1}(p_\alpha)$ is compact (and nonempty), and by (2.6), $\{P_\alpha, \Phi_{\alpha\beta}^*, P_\alpha, \mathcal{Q}\}$ is an inverse system. Its inverse limit is nonempty by (2.4) and is a subset of $e_\infty^{-1}((p_\alpha))$. Thus e_∞ is surjective and X_∞ is nonempty. Finally, X_∞ is HC since it is the θ -continuous image of the compact space X^* .

(2.9) *Let $\{X_\alpha, \Phi_{\alpha\beta}, \mathcal{Q}\}$ be an inverse system with surjective open bonding maps and surjective projections. If $\beta \in \mathcal{Q}$ and U is regular open in X_β , then $\langle U \rangle_\infty$ is regular open in X_∞ .*

PROOF. By (2.7), $\overline{\langle U \rangle_\infty} = \langle \bar{U} \rangle_\infty$. Suppose $(p_\alpha) \in \text{int} \langle \bar{U} \rangle_\infty$. Then $(p_\alpha) \in \langle V \rangle_\infty \subset \langle \bar{U} \rangle_\infty$ for some V open in some X_γ . Let $\delta > \gamma$ and $\delta > \beta$. Now since $\Phi_{\delta\gamma}$ is surjective, $\langle \Phi_{\delta\gamma}^{-1}(V) \rangle_\infty = \langle V \rangle_\infty$; also since the projections are surjective and $\langle \Phi_{\delta\gamma}^{-1}(V) \rangle_\infty \subseteq \langle \bar{U} \rangle_\infty$, $\Phi_{\delta\beta} \Phi_{\delta\gamma}^{-1}(V) \subseteq \bar{U}$. Thus $p_\beta \in \Phi_{\delta\beta} \Phi_{\delta\gamma}^{-1}(V)$

which, being open in $\bar{U}$, lies in U . Thus $(p_\alpha) \in \langle \Phi_{\delta\beta} \Phi_{\delta\gamma}^{-1}(V) \rangle_\infty \subset \langle U \rangle_\infty$, so $\text{int}(\bar{U})_\infty \subset \langle U \rangle_\infty$, and $\langle U \rangle_\infty$ is regular open.

(2.10) Let $\{X_\alpha, \Phi_{\alpha\beta}, \mathcal{Q}\}$ be an inverse system of nonempty compact sets and continuous surjections, let $\gamma \in \mathcal{Q}$, and let $p \in X_\gamma$. Then $\langle \{p\} \rangle_\infty \neq \emptyset$.

PROOF. Let $\delta \in \mathcal{Q}$ and let $\gamma, \delta < \varepsilon$. Put $P_\delta = \Phi_{\varepsilon\delta} \Phi_{\varepsilon\gamma}^{-1}(p)$. One sees that P_δ is well defined (i.e., independent of the choice of ε) and that $\{P_\alpha, \Phi_{\alpha\beta}|P_\alpha, \mathcal{Q}\}$ is an inverse system of nonempty sets. Thus by (2.4), $P_\infty \neq \emptyset$. By our selection of P_δ , $P_\infty = \langle \{p\} \rangle_\infty$ and the proof is complete.

(2.11) Let $\{X_\alpha, \Phi_{\alpha\beta}, \mathcal{Q}\}$ be an inverse system of nonempty HC spaces and continuous open surjections. Let $\gamma \in \mathcal{Q}$ and $p \in X_\gamma$. Then $\langle \{p\} \rangle_\infty \neq \emptyset$.

PROOF. Let $\{\theta X_\alpha, \Phi_{\alpha\beta}^*, \mathcal{Q}\}$ be the induced system of (2.6). Let $\bar{U}$ be any open ultrafilter on X_γ converging to p . By (2.10), $\langle \{U\} \rangle_\infty \neq \emptyset$ (in the induced system) and so $\emptyset \neq e_\infty(\langle \{U\} \rangle_\infty) \subset \langle \{p\} \rangle_\infty$.

(2.12) THEOREM. Let $\{X_\alpha, \Phi_{\alpha\beta}, \mathcal{Q}\}$ be an inverse system of MH spaces and continuous open surjections. Then X_∞ is MH.

PROOF. It is known that a space is MH if and only if it is HC and semiregular [K]. By (2.8), X_∞ is HC. It remains to show that X_∞ is semiregular. Let $(p_\alpha) \in X_\infty$ and let $\langle V \rangle_\infty$ be a neighborhood of (p_α) , where V is open in some X_γ . Since X_γ is MH, there exists a regular open set W such that $p_\gamma \in W \subset V$. We assert that $\langle W \rangle_\infty$ is regular open in X_∞ : Let $(q_\alpha) \in \text{int}(\bar{\langle W \rangle}_\infty)$. Then $(q_\alpha) \in \langle Z \rangle_\infty \subset \bar{\langle W \rangle}_\infty$ for some Z , open in some X_δ and, by (2.7), $\langle Z \rangle_\infty \subset \bar{\langle W \rangle}_\infty$. Let $\gamma, \delta < \varepsilon$. Then $\langle Z \rangle_\infty = \langle \Phi_{\varepsilon\delta}^{-1}(Z) \rangle_\infty \subset \bar{\langle W \rangle}_\infty$ implies that $Z_\gamma \equiv \Phi_{\varepsilon\gamma} \Phi_{\varepsilon\delta}^{-1}(Z) \subset \bar{W}$, for by (2.11), every $p \in \Phi_{\varepsilon\delta}^{-1}(Z)$ is the coordinate of some thread in X_∞ ; such a thread must have γ th coordinate in $\bar{W}$, so $\Phi_{\varepsilon\gamma}(p) \in \bar{W}$. Thus the open set Z_γ contains P_γ and is contained in $\bar{W}$. Since W is regular open, we have $Z_\gamma \subset W$. Now $q_\gamma \in Z \subset W$, so $(q_\alpha) \in \langle W \rangle_\infty$. We have shown that $\text{int}(\bar{\langle W \rangle}_\infty) \subset \langle W \rangle_\infty$; thus $\langle W \rangle_\infty$ is a regular open set in X_∞ containing (p_α) and lying entirely in $\langle V \rangle_\infty$. This completes the proof.

In (4.4) we give an example of a system of MH spaces with continuous surjection bonding maps with a nonminimal Hausdorff inverse limit.

3. HC spaces as the continuous open image of MH spaces. In this section, for any HC space X , we use Theorem (2.8) and construct an inverse system of HC spaces with continuous open surjective maps and with a MH inverse limit X_∞ that maps onto X . We first show that it is possible to "regularize" an open set U of X while retaining the H -closedness of X , i.e., we construct an HC space Z containing X in which U is a regular open subset of Z .

(3.1) *The regularization of an open set.* Let U be open in a space X and let h be a homeomorphism of X onto X' with $X \cap X' = \emptyset$. Let ψ be the natural map on $X \cup X'$ for the decomposition whose only nondegenerate elements are doubletons of the form $\{x, h(x)\}$ with $x \in X - U$. Denote by (X, U) the decomposition space with the quotient topology. Let $P: X \cup X' \rightarrow X$ be

given by $P(x) = x$, $x \in X$, and $P(x) = h^{-1}(x)$, $x \in X'$. P is easily seen to be continuous and open, and $P\psi^{-1} \equiv \pi$ is single-valued, continuous, open, and surjective. We call π the *natural projection* of (X, U) onto X .

One may think of (X, U) as the union of X with a disjoint copy U' of U so that the frontiers of U and U' coincide. Then U' forms a "blister" on X over U , causing both U and U' to be regular open in (X, U) . Specifically,

(3.2) U and $h(U)$ are regular open in (X, U) .

PROOF. Let $p \in [\text{cl}_{(X,U)} U] - U$. Since U and $h(U)$ are disjoint open in (X, U) , $p \notin h(U)$, so $p = \{q, h(q)\}$ for some $q \in X - U$. If W is open about p , then $\psi^{-1}(W)$ is open about $\{q, h(q)\}$ in $X \cup X'$. In particular, $\psi^{-1}(W)$ intersects $h(U)$, so $p \notin \text{int cl}_{(X,U)} U$. Therefore U is regular open, and similarly, so is $h(U)$.

(3.3) If X is HC and $U \subset X$ is open, then (X, U) is HC.

PROOF. This follows immediately since (X, U) is the continuous image of the HC space $X \cup X'$.

To follow is a process in which the open sets of X are allowed to "blister" one at a time, cumulatively. The resulting inverse limit space will be semiregular and will map openly onto X .

(3.4) Let S be a well-ordered set with no supremum. Then there is a well-ordered set T and a surjective function $G: T \rightarrow S$ with cofinal point inverses.

PROOF. Let $T = S \times S$ be ordered by: $(a, b) < (c, d)$ if $a < c$ or if $a = c$ and $b < d$. Then T is well ordered. Put $G(a, b) = b$. Then $G^{-1}(b) = \{(a, b) | a \in S\}$ which is cofinal in T .

(3.5) CONSTRUCTION. Let X_1 be HC with topology $\mathfrak{B}$. Let $\mathfrak{B}$ be well ordered with no supremum. By (3.4) there is a well-ordered set $\mathcal{Q}$ and a surjective indexing function $i: \mathcal{Q} \rightarrow \mathfrak{B}$ with cofinal point inverses. Denote by U_α the image $i(\alpha)$. Then for each $B \in \mathfrak{B}$ and $\alpha \in \mathcal{Q}$ there exists $\beta > \alpha$ such that $B = U_\beta$. We define an inverse system $\{X_\alpha, \Phi_{\alpha\beta}, \mathcal{Q}\}$ inductively as follows: let $X_2 = (X_1, U_1)$ and let $\Phi_{21}: X_2 \rightarrow X_1$ be the natural projection. Suppose for $\delta \leq \gamma \leq \beta < \alpha$ that X_β and $\Phi_{\beta\gamma}: X_\beta \rightarrow X_\gamma$ are defined with X_β HC, $\Phi_{\beta\gamma}$ continuous, open and surjective, and $\Phi_{\beta\delta} = \Phi_{\gamma\delta} \Phi_{\beta\gamma}$. For $\alpha \in \mathcal{Q}$, denote by $\mathcal{Q}_\alpha$ the set $\{\beta \in \mathcal{Q} | \beta < \alpha\}$. We distinguish two cases. If α is a limit ordinal, let $X_\alpha = \text{inv lim}\{X_\beta, \Phi_{\beta\gamma}, \mathcal{Q}_\alpha\}$. Let $\Phi_{\alpha\beta}: X_\alpha \rightarrow X_\beta$ be the projection onto the β th coordinate. If α is not a limit ordinal, let $X_\alpha = (X_{\alpha-1}, \Phi_{(\alpha-1)}^{-1}(U_{\alpha-1}))$ and let $\Phi_{\alpha(\alpha-1)}: X_\alpha \rightarrow X_{\alpha-1}$ be the natural projection with $\Phi_{\alpha\beta} = \Phi_{(\alpha-1)\beta} \Phi_{\alpha(\alpha-1)}$.

For $\beta < \alpha$ let $\Phi_{\alpha\beta}^*: \theta X_\alpha \rightarrow \theta X_\beta$ be the induced map of (2.6). Let

$$X^* = \text{inv lim}\{\theta X_\alpha, \Phi_{\alpha\beta}^*, \mathcal{Q}\}.$$

Let $e_\alpha: \theta X_\alpha \rightarrow X_\alpha$ and $e_\infty: X^* \rightarrow X_\infty$ be the evaluation maps of (2.6) and (2.8).

NOTATION. Unless otherwise defined, objects referred to in the balance of this section are those of the above construction.

(3.6) The maps $\Phi_{\alpha\beta}$ are continuous open surjections.

PROOF. Suppose $\Phi_{\beta\gamma}$ is a continuous open surjection for all $\gamma \leq \beta < \alpha$.

Case 1. α is a limit ordinal.

We make the inductive assumption that for $\beta < \alpha$ and $p_\beta \in X_\beta$, there is a thread (p_t) , $1 \leq t \leq \beta$, so that if $1 \leq s \leq t \leq \beta$, then the consistency property $\Phi_{ts}(p_t) = p_s$ holds. Let $\beta \in \mathcal{Q}$ and $p_\beta \in X_\beta$. Thus, for each $1 \leq t \leq \beta$, assume we have chosen $p_t \in X_t$ so that the thread (p_t) has the above consistency property.

Suppose $\beta < \delta < \alpha$ and for all $\beta \leq \gamma < \delta$ we have chosen $p_\gamma \in X_\gamma$ so that the thread (p_t) , $1 \leq t \leq \gamma$, is a thread with the consistency property. If $\delta = \delta_0 + 1$, just choose $p_\delta \in X_\delta$ so that $\Phi_{\delta\delta_0}(p_\delta) = p_{\delta_0}$. If δ is a limit ordinal, let $p_\delta = (p_t)$, $0 \leq t < \delta$. Finally, p_α is defined to be (p_t) , $0 \leq t < \alpha$. Therefore each $\Phi_{\alpha\beta}$ is a surjection.

For $B \subset X_\beta$, $\beta < \alpha$, define $\langle B \rangle_\alpha = \{(p_\delta) \in X_\alpha \mid p_\beta \in B\}$. If U is open in X_β , then $\Phi_{\alpha\beta}^{-1}(U) = \langle U \rangle_\alpha$ which is basic open in X_α by (2.3), so $\Phi_{\alpha\beta}$ is continuous. Conversely, if $\langle U \rangle_\alpha$ is basic open in X_α , where $U \subset X_\delta$ for some $\delta \in \mathcal{Q}_\alpha$, then $\Phi_{\alpha\beta}(\langle U \rangle_\alpha) = \Phi_{\beta\delta}^{-1}(U)$ or $\Phi_{\delta\beta}(U)$ according as $\delta < \beta$ or $\beta \leq \delta$, open in either case. Thus $\Phi_{\alpha\beta}$ is open.

Continuity comes from the fact that the functions are coordinate projections from the inverse limit.

Case 2. α is not a limit ordinal. $\Phi_{\alpha(\alpha-1)}$ is a continuous open surjection as noted earlier, as is $\Phi_{(\alpha-1)\beta}$ by inductive hypothesis. Thus so is $\Phi_{\alpha\beta} = \Phi_{(\alpha-1)\beta}\Phi_{\alpha(\alpha-1)}$.

(3.7) The projection $\Pi: X_\infty \rightarrow X_1$ is a continuous open surjection.

PROOF. Apply the argument of Case 1 above.

(3.8) X_α is HC for each $\alpha \in \mathcal{Q}$.

PROOF. Suppose true for $\beta < \alpha$. If α is not a limit ordinal, then $X_\alpha = (X_{\alpha-1}, \Phi_{(\alpha-1)\alpha}^{-1}(U_{\alpha-1}))$ which is HC by (3.3). Otherwise

$$X_\alpha = \text{inv lim} \{X_\beta, \Phi_{\beta\gamma}, \mathcal{Q}_\alpha\}$$

which is HC by (2.8).

(3.9) X_∞ is semiregular.

PROOF. Let $(p_\alpha) \in X_\infty$ and let R be an open subset of X_∞ containing (p_α) . By (2.3) there exists $\beta \in \mathcal{Q}$ and V open in X_β such that $(p_\alpha) \in \langle V \rangle_\infty \subset R$. Now $\varphi_{\beta 1}(V) \in B$ and, as seen in (3.5), we may assume that $\varphi_{\beta 1}(V) = U_\gamma$ where $\gamma > \beta$. Note that since the bonding maps are surjections,

$$\begin{aligned} \langle V \rangle_\infty &= \langle \varphi_{\gamma\beta}^{-1}(V) \rangle_\infty = \langle \varphi_{\gamma 1}^{-1}\varphi_{\beta 1}(V) \rangle_\infty = \langle \varphi_{\gamma 1}^{-1}(U_\gamma) \rangle_\infty \\ &= \langle \varphi_{(\gamma+1)\gamma}^{-1}[\varphi_{\gamma 1}^{-1}(U_\gamma)] \rangle_\infty. \end{aligned}$$

Now by (3.2), $\varphi_{(\gamma+1)\gamma}^{-1}[\varphi_{\gamma 1}^{-1}(U_\gamma)]$ is the union of two disjoint regular open sets. Let T be the one containing $p_{\gamma+1}$. Then by (2.9), $\langle T \rangle_\infty$ is regular open and $(p_\alpha) \in \langle T \rangle_\infty \subset \langle V \rangle_\infty \subset R$ and X_∞ is semiregular.

Thus we have proved the following:

(3.10) THEOREM. Every HC space is the continuous open image of a MH space.

PROOF. X_∞ is an HC, semiregular space that is mapped onto X by the projection $\pi: X_\infty \rightarrow X_1$.

4. Examples.

(4.1) Let $\tilde{X}$ be the set of points in the plane of the form $a_{mn} = (1/m, 1/n)$ or $c_m = (1/m, 0)$, where m and n are positive integers. Give X the relative usual topology. Suppose $\alpha \notin \tilde{X}$ and let $X = \tilde{X} \cup \{\alpha\}$. Basic neighborhoods of α are sets of the form $\{a_{mn} | n > n_0\} \cup \{\alpha\}$. Then X is HC but not MH (X is not semiregular at α). Let $U = X - \{c_m\}_{m=1}^\infty$. One checks that (X, U) is MH and maps openly onto X by the natural projection.

(4.2) Let Y be the closed interval $[0, 1]$ and let the topology be generated by the usual one together with the set Q of rational numbers in $[0, 1]$. Then Y is HC but not semiregular at any rational number. However, (Y, Q) is MH and maps onto Y .

(4.3) Let X be the space defined in (4.1) and let $X_n = X$ for $n = 1, 2, \dots$. Define $\Phi_{n(n-1)}: X_n \rightarrow X_{n-1}$ by $a_{mn} \rightarrow a_{(m+1)n}$, $c_m \rightarrow c_m$, and $\alpha \rightarrow \alpha$. Consider $\{X_n, \Phi_{nn}, I_+\}$ where I_+ is the set of positive integers and

$$\Phi_{mn} = \Phi_{(n+1)n} \Phi_{(n+2)(n+1)} \cdots \Phi_{m(m-1)}.$$

The only threads in the system have the form $(c_n, c_n, c_n, \dots)$ or $(\alpha, \alpha, \alpha, \dots)$, and we see that X_∞ is an infinite discrete space, hence not HC. Thus the requirement in (2.8) that the maps $\Phi_{\alpha\beta}$ be surjective is necessary.

(4.4) The requirement in (2.12) that $\Phi_{\alpha\beta}$ be open is necessary. For let Y be the space (X, U) of (4.1). We "unzip" Y along $X - U$ in the following sense to get two disjoint open copies of X . Let $\tilde{Y}_k$ consist of the planar points $a_{mn} = (1/m, 1/n)$, $c_m = (1/m, 0)$ ($m, n \in I_+$), $d_m = (1/m, -1)$ ($m \leq k$), and $b_{mn} = (1/m, 1/-n - 1)$ for $m \leq k$ and $b_{mn} = (1/m, 1/-n)$ for $m > k$. Adjoin to $\tilde{Y}_k$ the two new points α, β whose basic neighborhoods have the form $\{a_{mn} | n > n_0\} \cup \{\alpha\}$ and $\{b_{mn} | n > n_0\} \cup \{\beta\}$, respectively, where $n_0 \in I_+$. Call this augmented space Y_k . Define $\Phi_{k(k-1)}: Y_k \rightarrow Y_{k-1}$ to be the identity function except that $\Phi_{k(k-1)}(d_k) = c_k$. Note that Y_k is MH and $\Phi_{k(k-1)}$ is continuous and surjective but not open. For $j < k$, put $\Phi_{kj} = \Phi_{(j+1)j} \cdots \Phi_{k(k-1)}$ as usual and let Φ_{kk} be the identity function on Y_k . Consider $Y_\infty = \text{inv lim}\{Y_k, \Phi_{kj}, I_+\}$. We observe that its elements are of the following form.

- (1) $\bar{\alpha} = (\alpha, \alpha, \alpha, \dots)$,
- (2) $\bar{\beta} = (\beta, \beta, \beta, \dots)$,
- (3) $\bar{a}_{mn} = (a_{mn}, a_{mn}, \dots)$,
- (4) $\bar{b}_{mn} = (b_{mn}, b_{mn}, \dots)$,
- (5) $\bar{c}_m = (c_m, c_m, c_m, \dots)$,
- (6) $\bar{d}_m = (c_m, c_m, \dots, c_m, d_m, d_m, \dots)$ ($m-1$ c_m 's).

It is easy to see that $\bar{a}_{mn}$ and $\bar{b}_{mn}$ are isolated (e.g., for any $k \in I_+$ let $a_{mn} \in Y_k$; since $\{a_{mn}\}$ is open in Y_k , $\{\bar{a}_{mn}\} = \langle \{a_{mn}\} \rangle_\infty$ is open). Also, basic neighborhoods of $\bar{\alpha}$ and $\bar{\beta}$ are $\{\bar{a}_{mn} | n > n_0\} \cup \{\bar{\alpha}\}$ and $\{\bar{b}_{mn} | n > n_0\} \cup \{\bar{\beta}\}$, respectively. If V is a basic neighborhood of c_m in Y_k , then

$$V = \{a_{mn} | n > n_0\} \cup \{c_m\}$$

or

$$V = \{a_{mn} | n > n_0\} \cup \{b_{mn} | n > n_0\} \cup \{c_m\}$$

according as $k \geq m$ or $k < m$. The former case provides the basic neighborhoods $\langle V \rangle_\infty$ of $\bar{c}_m$; namely, $\{\bar{a}_{mn} | n > n_0\} \cup \{\bar{c}_m\}$. Analogously, basic neighborhoods of $\bar{d}_m$ are of the form $\langle W \rangle_\infty$, where W is a basic neighborhood of d_m in Y_k , $k \geq m$, and $\langle W \rangle_\infty = \{\bar{b}_{mn} | n > n_0\} \cup \{\bar{d}_m\}$. Now $Y = \{\bar{a}_{mn}, \bar{b}_{mn}, \bar{c}_m, \bar{d}_m, \bar{\alpha}, \bar{\beta}\}$ is the disjoint open union of $\{\bar{a}_{mn}, \bar{c}_m, \bar{\alpha}\}$ and $\{\bar{b}_{mn}, \bar{d}_m, \bar{\beta}\}$, each copies of the space X of (4.1). Hence Y_∞ fails to be semiregular at $\bar{\alpha}$ and $\bar{\beta}$, and so is not MH.

ADDENDUM. In a private communication, the authors have learned that L. M. Friedler and D. H. Pethey have independently solved Willard's conjecture using inverse limit techniques. In this paper we utilized the induced system of absolutes over a system with open surjective bonding maps, while in their result, Friedler and Pethey proved that the inverse limit of a system of nonempty HC spaces over a directed set with a cofinal chain is nonempty and HC. It is not known whether the inverse limit of nonempty HC spaces with surjective bonding maps is nonempty and HC.

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