MODULES AND QUADRATIC FORMS OVER POLYNOMIAL ALGEBRAS

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ABSTRACT. Let D be a finite dimensional division algebra over a field k . Projective $D[x, y]$ -modules of rank ≥ 2 are free. Projective ideals of $D[x, y]$, D a quaternionic division algebra, which are not free, are used to construct regular quadratic $k[x, y]$ -modules which are not extended from k .

1. Projective modules.

PROPOSITION 1.1. *Let D be a finite dimensional division algebra over a field k . Let $R = k[x, y]$ be the polynomial algebra in two variables and $A = D[x, y]$. Then every finitely generated projective A -module of rank ≥ 2 is free.*

PROOF. Let $S = k[x]$, $C = D[x]$ and for each maximal ideal m of S , $C_m = S_m \otimes_S C$. Let P be a finitely generated projective A -module with $n = \text{rank } P \geq 2$. Since $K_0(A) = K_0(D)$, P is stably free. The maximal spectrum of $S_m[y]$ is the union of two 1-dimensional sets [2, III, Proposition 3.13], hence by the cancellation theorem of Bass and Schanuel [2, IV, Theorem 3.1], $C_m \otimes_C P \cong C_m[y]^n$. In other words, the $C[y]$ -module P is locally extended from C . By a slight generalisation of Quillen's theorem [10, Theorem 1] P is extended from C , i.e. $P \cong C[y] \otimes_C Q$ for some C -module Q . Since projective C -modules are free, so is P .

REMARK 1.2. Examples of nonfree projective ideals in $D[x, y]$ are given in [8].

2. Quadratic modules. Let R be a commutative ring, V a finitely generated R -module and $f: V \times V \rightarrow R$ a bilinear symmetric pairing. We call the pair (V, f) a regular quadratic R -module if the map of V into $\text{Hom}_R(V, R)$ induced by f is an isomorphism. If (V, f) is a regular quadratic R -module and S is any commutative R -algebra, then $(S \otimes_R V, 1_S \otimes f)$ is a regular quadratic S -module. A regular quadratic S -module is *extended from R* if it is isomorphic to some $(S \otimes_R V, 1_S \otimes f)$. Let $R = k[t_1, \dots, t_n]$ be a polynomial algebra in n variables over a field k of characteristic not 2. For $n = 1$, Harder proved that regular quadratic R -modules are always extended from k (see [5, Theorem 13.4.1]). In [9] S. Parimala proved that regular quadratic R -modules of rank 2 are always extended, and constructed

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infinitely many rank 4 quadratic modules over $\mathbf{R}[x, y]$ which are not extended. Very recently M. S. Raghunathan proved that regular quadratic R -modules are extended if the Brauer group $\text{Br}(k)$ of k has no element of order two [11]. Here we show that this result is wrong in general without the condition on $\text{Br}(k)$.

PROPOSITION 2.1. *Let k be a field of characteristic not 2 or 3 over which there exists a quaternionic division algebra D . There exist two regular quadratic $k[x, y]$ -modules of rank 3 (resp. 4) which are not extended from k .*

PROOF. We first construct a central separable $k[x, y]$ -algebra which is not extended from k . Put $R = k[x, y]$ and $A = D[x, y]$. According to [8], A contains a nonfree projective left ideal I and the central separable R -algebra $B = \text{End}_A(I)$ is not isomorphic to A . Clearly $A/(x, y)A \cong B/(x, y)B \cong D$ as k -algebras, hence if B were extended, then it would be isomorphic to A . We now define quadratic forms on A and B . The reduced trace and the reduced norm of D induce, by scalar extension, the reduced trace T_A and the reduced norm N_A of A . By general results on central separable algebras [3], [6], B also has a reduced trace T_B and a reduced norm N_B . Since A and B are free of rank four over R , N_A and N_B define, in the usual way, quadratic structures on A and B . The quadratic modules A and B are easily seen to be regular (for instance by tensoring over k with the algebraic closure of k and using faithfully flat descent). Now notice that, if B is extended from k , then B must be isometric to A (go modulo (x, y) and use the fact that the reduced norm "commutes" with scalar extension). Following N. Jacobson [4, p. 185] and using a result of Ancochea [1], we show that $A \cong B$ as quadratic modules implies an isomorphism of algebras, contradicting the result obtained above. Let $\prime: A \rightarrow B$ be an isomorphism of R -modules such that $N_B(a') = N_A(a)$ for every element a in A . In particular, $1 = N_A(1) = N_B(1')$. Hence, $1'$ is a unit of B , and replacing $a \mapsto a'$ by $a \mapsto a'(1')^{-1}$, we can assume that $1' = 1$. By linearity $r' = r$ for every r in R . Then, for any a in A ,

$$\begin{aligned} N_B((r + a)') &= N_B(r' + a') = N_B(r + a') \\ &= r^2 + rT_B(a') + N_B(a'), \\ N_A(r + a) &= r^2 + rT_A(a) + N_A(a). \end{aligned}$$

Since R is infinite this implies that $T_A(a) = T_B(a')$. From this, we first deduce that $\prime$ is a semi-isomorphism in the sense of [1], i.e. $(ab + ba)' = a'b' + b'a'$. Then, using the fact that $A \otimes_R k(x, y) \cong B \otimes_R k(x, y)$, one can conclude, as in [1], that $\prime$ is an isomorphism or an anti-isomorphism. Since A is isomorphic to its opposite, A and B are isomorphic. Since $(s, t) \mapsto T_B(st)$ is a regular pairing, in order to show that $(ab + ba)' = a'b' + b'a'$, it suffices to show that, for every c in A , $T_B((ab + ba)'c') = T_B((a'b' + b'a')c')$. We first claim that, for every a in A ,

$$(1) \quad T_A(a^2) = T_B((a')^2),$$

$$(2) \quad T_A(a^3) = T_B((a')^3).$$

In fact a and a' satisfy the Cayley-Hamilton polynomials [3], [6]

$$(3) \quad a^2 - aT_A(a) + N_A(a) = 0,$$

$$(4) \quad (a')^2 - a'T_B(a') + N_B(a') = 0.$$

Taking traces of (3) and (4) we get (1). Multiplying (3) by a , (4) by a' and using (1) we get (2).

Now, since $T_B(st) = T_B(ts)$, it is easily checked that

$$2T_B((ab)'c') = T_B(((ab) + c)')^2 - ((ab)')^2 - (c')^2).$$

Using (1) we then get $T_B((ab)'c') = T_A(abc)$. Similarly, from

$$\begin{aligned} 3T_B((a'b' + b'a')c') \\ = T_B(((a + b + c)')^3 - ((a + b)')^3 - ((a + c)')^3 \\ - ((b + c)')^3 + (a')^3 + (b')^3 + (c')^3) \end{aligned}$$

and from (2) we get $T_B((a'b' + b'a')c') = T_A((ab + ba)c)$. This shows that $'$ is a semi-isomorphism of R -algebras.

REMARK 2.1. Since $N_B(1) = 1$, the submodule R of B has an orthogonal complement B_0 of rank 3. Since B is not extended from k , neither is B_0 . This is in contrast with the result of S. Parimala [9] asserting that quadratic modules of rank 2 are always extended from k . Notice that B_0 must be indecomposable.

REMARK 2.2. The Clifford algebras C_A and C_B of the quadratic modules A and B are both isomorphic to $M_2(A)$. This can be shown in the following way. Since A and B are stably isometric, C_A and C_B (which are both central separable R -algebras) are Brauer equivalent over R . Hence $M_n(C_A) \cong M_n(C_B)$ for some n . According to [8], to prove $C_A \cong C_B$, it suffices to show that, for any left C_A -module I , $I^n \cong C_A^n$ implies that I is a bimodule. We prove that any such I is in fact free. Observe that $C_A \cong M_2(A)$ [7, p. 116, Corollary 3.3]. Hence by Morita theory, we only have to show that $A^2 \otimes_{C_A} I$ is free as left A -module. This is a special case of Proposition 1.1.

We thank A. Geramita for suggesting the following proposition, which shows that there exist nonextended quadratic $\mathbf{R}[x, y]$ -modules of arbitrarily large rank.

PROPOSITION 2.3. *Let B_0 be the quadratic $\mathbf{R}[x, y]$ -module of Remark 2.1 (with $k = \mathbf{R}$). Let M be the orthogonal direct sum of n copies of B_0 . Then M is not extended from $\mathbf{R}$.*

PROOF. Let $f(X_1, X_2, X_3)$ be the quadratic form with coefficients in $\mathbf{R}[x, y]$ corresponding to B_0 . Observe that for any real maximal ideal $m = (x - \xi, y - \eta)$ of $\mathbf{R}[x, y]$ the $\mathbf{R}$ -algebra B/mB is the usual quaternion algebra. Hence replacing x, y by ξ, η in the coefficients of $f(X_1, X_2, X_3)$ yields a positive definite quadratic form over $\mathbf{R}$. The quadratic form corresponding to M is

$f(X_{11}, X_{12}, X_{13}) + \cdots + f(X_{n1}, X_{n2}, X_{n3})$, and it suffices to show that it does not represent 1 over $\mathbf{R}[x, y]$. Suppose that, for some $U_{ij} \in \mathbf{R}[x, y]$,

$$F(x, y) = f(U_{11}, U_{12}, U_{13}) + \cdots + f(U_{n1}, U_{n2}, U_{n3}) = 1.$$

Then, for any $\xi, \eta \in \mathbf{R}$, $F(\xi, \eta) = 1$ and each $f(U_{i1}(\xi, \eta), U_{i2}(\xi, \eta), U_{i3}(\xi, \eta))$, being nonnegative, must lie between 0 and 1. Hence all the $f(U_{i1}(x, y), U_{i2}(x, y), U_{i3}(x, y))$ are constant and at least one of them is $\neq 0$. This is impossible because B_0 is indecomposable and therefore $f(X_1, X_2, X_3)$ does not represent a nonzero constant over $\mathbf{R}[x, y]$.

ADDED IN PROOF. The argument in 2.3 can be refined to give examples of nonextended quadratic modules of any rank ≥ 3 .

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