THE IMAGE OF G BORDISM IN Z_2 BORDISM

R. PAUL BEEM

ABSTRACT. For finite cyclic groups G of even order, the image of unoriented G bordism in $Z/2Z$ bordism and the kernel of the extension homomorphism from $Z/2Z$ to G bordism depend only on whether or not the order of G is divisible by four. If so, then these sets are equal and are equal to the image of circle bordism in $Z/2Z$ bordism and the kernel of extension to circle bordism. If not, then extension is a monomorphism and restriction is an epimorphism.

The purpose of this note is to show that for finite cyclic groups G of even order the image of unoriented G bordism in $Z/2Z$ bordism and the kernel of the extension homomorphism from $Z/2Z$ to G bordism depend only on whether or not the order of G is divisible by four. If so, then these sets are equal and are equal to the image of circle bordism in $Z/2Z$ bordism and the kernel of extension to circle bordism. If not, then extension is a monomorphism and restriction is an epimorphism.

We will use the definitions and notational conventions of [2]. Recall from [1] that there is an action, defined via the twisted product of G spaces,

$$N_*(G; \text{Free}) \otimes N_*(G; \mathfrak{F}, \mathfrak{F}') \rightarrow N_*(G; \mathfrak{F}, \mathfrak{F}'),$$

which makes $N_*(G; \mathfrak{F}, \mathfrak{F}')$ into an algebra over the Hopf algebra $N_*(G; \text{Free})$. The degree one endomorphism of $N_*(Z_2; \text{All})$ given by twisting with the class of the circle with antipodal involution is a derivation and will be denoted by θ . In [1] it was shown that the kernel of this derivation is precisely the image of circle bordism.

THEOREM. Suppose $Z_2 \subset G \subset S^1$, G a finite group. If $Z_4 \subset G$, then the following are equal in $N_*(Z_2; \text{All})$:

- (A) kernel of extension $e(G): N_*(Z_2; \text{All}) \rightarrow N_*(G; \text{All})$;
- (B) kernel of extension $e(S^1): N_*(Z_2; \text{All}) \rightarrow N_*(S^1; \text{All})$;
- (C) kernel of $\theta: N_*(Z_2; \text{All}) \rightarrow N_*(Z_2; \text{All})$;
- (D) image of restriction $r(S^1): N_*(S^1; \text{All}) \rightarrow N_*(Z_2; \text{All})$;
- (E) image of restriction $r(G): N_*(G; \text{All}) \rightarrow N_*(Z_2; \text{All})$.

PROOF. Clearly $A \subset B$ and $D \subset E$. Since $((S^1 \times M)/(-1 \times T), 1 \times T) = r(S^1) \circ e(S^1)(M, T)$, where T denotes an involution on M , $B \subset C$. $C = D$ is shown in [1].

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Suppose x is in $N_*(G; \text{All})$. Then $e(G) \circ r(G)(x) = (G/Z_2, G)x$; see [2]. Since G/Z_2 has even order, $(G/Z_2, G)$ bounds in $N_0(G; \text{All})$, so that $(G/Z_2, G)x$ bounds in $N_*(G; \text{All})$. See [2, p. 68]. Therefore $E \subset A$. This finishes the proof.

If $G = Z_{2n}$, where n is odd, then $e(G)(M, T) = (M \times Z_n, \phi)$, where $\phi: G \times (M \times Z_n) \rightarrow (M \times Z_n)$ is given by

$$\begin{aligned}\phi\left(t, \left(m, (t^2)^p\right)\right) &= \left(m, (t^2)^{p+1}\right), & p < n-1, \\ &= (Tm, t^{2n}), & p = n-1,\end{aligned}$$

where t generates Z_{2n} . Restricting this to a Z_2 action gives n copies of (M, T) which is bordant to (M, T) . Hence, $e(G)$ is monic and $r(G)$ is epic.

REFERENCES

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DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, SOUTH BEND, INDIANA 46615