STATES ON QUANTUM LOGICS AND THEIR CONNECTION WITH A THEOREM OF ALEXANDROFF

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ABSTRACT. We generalize the notion of regularity of measures to quantum logics and then prove that each regular finitely additive state on a quantum logic is countably additive. Examples are given from measure theory and quantum mechanics.

In this note, we establish a generalization of the well-known theorem of Alexandroff: Each bounded, regular and finitely additive measure on a compact Hausdorff space is countably additive [2, p. 138].

Let $(\mathcal{L}, \leq)$ be a partially ordered set with at least two elements, a largest element e and a smallest element 0 . Let $\mathcal{L}$ be equipped with an orthocomplementation $'$. Specifically, if $p \in \mathcal{L}$ then $p' \in \mathcal{L}$ and the following are satisfied: (i) $(p')' = p$ for all $p \in \mathcal{L}$; (ii) $p \leq q$ in $\mathcal{L}$ implies $q' \leq p'$; (iii) $p \vee p' = e$ and $p \wedge p' = 0$ for all $p \in \mathcal{L}$. We say that p, q are *orthogonal*, denoted $p \perp q$, if $p \leq q'$. We will assume that if $\{p_1, p_2, \dots, p_n\}$ is a finite orthogonal set in $\mathcal{L}$ then $\bigvee_{k=1}^n p_k$ exists in $\mathcal{L}$. Finally, we insist that $(\mathcal{L}, \leq)$ satisfies the *orthomodular identity*: $p \leq q$ in $\mathcal{L}$ implies $q = p \vee (q \wedge p')$. A set $\mathcal{L}$ satisfying all the conditions above will be called a *quantum logic* and its elements *propositions*. $(\mathcal{L}, \leq)$ is called *σ -orthocomplete* if for each countable orthogonal sequence $\{p_1, p_2, \dots\}$ in $\mathcal{L}$ there exists $p \in \mathcal{L}$ such that $p = \bigvee_{k=1}^{\infty} p_k$. We note here that a quantum logic need not be a lattice.

A *state* $\omega: \mathcal{L} \rightarrow [0, 1]$ is a function with the properties: (i) $\omega(0) = 0$ and $\omega(e) = 1$; (ii) if $\{p_1, p_2, \dots, p_n\}$ is a finite orthogonal set in $\mathcal{L}$ with $p = \bigvee_{k=1}^n p_k$, then $\omega(p) = \sum_{k=1}^n \omega(p_k)$. A state ω is called *countably additive* if $\{p_1, p_2, \dots\}$ is an orthogonal sequence with $p = \bigvee_{k=1}^{\infty} p_k$ in $\mathcal{L}$ then $\omega(p) = \sum_{k=1}^{\infty} \omega(p_k)$. Let $\mathcal{P} \subseteq \mathcal{L}$; a state ω is called *$\mathcal{P}$ -regular* if for each $\varepsilon > 0$ and each $q \in \mathcal{L}$ there exists $p \in \mathcal{P}$ with $p \leq q$ and $\omega(q \wedge p') < \varepsilon$. If ω is $\mathcal{P}$ -regular then for each $\varepsilon > 0$ and $q \in \mathcal{L}$ there exists $p' \in \mathcal{L}$ for some $p \in \mathcal{P}$ such that $p' \geq q$ and $\omega(p' \wedge q') < \varepsilon$. (There exists $p \in \mathcal{P}$ such that $p \leq q'$ and $\omega(p' \wedge q') < \varepsilon$.)

Let $\mathcal{P}' = \{p': p \in \mathcal{P}\}$. An element $p \in \mathcal{P}$ is called *finitely-coverable* if $\{p'_1, p'_2, \dots\} \subseteq \mathcal{P}'$, $\bigvee_{k=1}^{\infty} p'_k$ exists in $\mathcal{L}$ and $p \leq \bigvee_{k=1}^{\infty} p'_k$ implies for some n that $\bigvee_{k=1}^n p'_k$ exists in $\mathcal{L}$ and $p \leq \bigvee_{k=1}^n p'_k$. $\mathcal{P}$ is called *finitely-coverable* if each element of $\mathcal{P}$ is finitely-coverable. We now state the

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generalized form of Alexandroff's theorem.

THEOREM. *Suppose $\mathcal{L}$ has the following properties: $\mathcal{L}$ is a σ -orthocomplete quantum logic; $\mathcal{P} \subseteq \mathcal{L}$ is finitely-coverable; $\mathcal{P}'$ contains the join of any sequence in $\mathcal{P}'$ and ω is a $\mathcal{P}$ -regular state on $\mathcal{L}$. Then ω is countably additive on $\mathcal{L}$.*

PROOF. Let $\{q_1, q_2, \dots\}$ be an orthogonal sequence in $\mathcal{L}$ with $q = \bigvee_{k=1}^{\infty} q_k$ in $\mathcal{L}$. Then $q \geq \bigvee_{k=1}^n q_k$ for all n and $1 \geq \omega(q) \geq \sum_{k=1}^n \omega(q_k)$. Hence, $\omega(q) \geq \sum_{k=1}^{\infty} \omega(q_k)$. Let $\varepsilon > 0$ be given; then there exists $\{p_1, p_2, \dots\} \subseteq \mathcal{P}$ with $p'_k \geq q_k$ and $\omega(q'_k \wedge p'_k) < \varepsilon/2^k$. Also, there exists $p \in \mathcal{P}$ with $p \leq q$ and $\omega(q \wedge p') < \varepsilon$. Since p is finitely-coverable, there exists m such that $p \leq \bigvee_{k=1}^m p'_k$. From the orthomodularity of $\mathcal{L}$ and finite additivity of ω , $\omega(p'_k) = \omega(q_k) + \omega(q'_k \wedge p'_k)$, and this implies $\omega(p'_k) - \varepsilon/2^k \leq \omega(q_k)$. Hence, $\sum_{k=1}^{\infty} \omega(q_k) \geq \sum_{k=1}^{\infty} \omega(p'_k) - \varepsilon \geq \sum_{k=1}^m \omega(p'_k) - \varepsilon \geq \omega(p) - \varepsilon \geq \omega(q) - 2\varepsilon$. Therefore, $\omega(q) = \sum_{k=1}^{\infty} \omega(q_k)$.

We now give two examples. First, let X be a compact Hausdorff space; identify $\mathcal{L}$ with the σ -algebra of Borel sets of X and let $\mathcal{P}$ be the closed sets of X . Since $\mathcal{P}'$ consists of the open sets, $\mathcal{P}'$ is closed under countable joins and $\mathcal{P}$ is finitely-coverable. If ω is a finitely additive, regular, probability measure on $\mathcal{L}$ then ω is countably additive.

Second, let $\mathcal{L}$ be the σ -complete orthomodular lattice of projections on a separable Hilbert space and let $\mathcal{P}$ be the set of all projections each of which has finite dimensional range, see [1] or [3]. Each element $p \in \mathcal{L}$ (not only those in $\mathcal{P}$) are finitely-coverable. To see this let $p \in \mathcal{L}$ and let $\{p'_1, p'_2, \dots\} \in \mathcal{P}'$ such that $p \leq \bigvee_{k=1}^{\infty} p'_k$. Each p'_k has as its range a subspace with finite co-dimension. If the integer $n_k, k = 1, 2, \dots$ is the co-dimension of the range of $p'_1 \vee \dots \vee p'_k$ then $n_1 \geq n_2 \geq \dots \geq 0$. Thus, for some $m, n_m = n_{m+1} = \dots$ and $p \leq \bigvee_{k=1}^m p'_k$. If ω is a $\mathcal{P}$ -regular state then ω is countably additive. The countably additive states in this case have been identified by Gleason, see, e.g., [3, p. 132] as the positive trace class operators of unit trace. The fact that each such state is $\mathcal{P}$ -regular follows from the spectral theorem. Let $\omega = \sum_n \lambda_n e_n \otimes \bar{e}_n$ and $p = \sum_k f_k \otimes \bar{f}_k$ where (e_n) and (f_k) are orthonormal sequences and $\lambda_n > 0$ with $\sum_n \lambda_n = 1$, see [1]. Then $\omega(p) = \text{tr}(\omega \circ p) = \sum_{n,k} \lambda_n |(e_n, f_k)|^2 \leq 1$. For $\varepsilon > 0$ there exists m sufficiently large for which $q = \sum_{k=1}^m f_k \otimes \bar{f}_k$ has the property:

$$\omega(p \wedge q') = \omega(p) - \omega(q) = \sum_{\substack{k > m \\ n}} \lambda_n |(e_n, f_k)|^2 < \varepsilon.$$

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