CLASS GROUPS OF RINGS OF INVARIANTS¹

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ABSTRACT. Let G be a connected algebraic group acting on a Krull domain S with ring of invariants R . Then an extension E of $\text{Cl}(R)$ by a free abelian group (related to blowing up) maps to $\text{Cl}(S)$ with kernel related to characters of G .

This paper studies the class groups of the algebras of invariants under affine group actions. For finite group schemes the theory is familiar [2, Chapter IV], so we assume the group is connected. For algebraic groups over algebraically closed fields the main result was proved by Magid [3]; the argument here is similar in outline but requires more Hopf-algebraic techniques.

1. Hypotheses and notations. Let k be a field. Let $G = \text{Spec}(A)$ be an affine group scheme over k , so A is a commutative Hopf algebra. We assume that k is absolutely algebraically closed in A , or equivalently [5, p. 264] that $A \otimes \bar{k}$ has no nontrivial nilpotents or idempotents. If A is finitely generated, this means that G is a connected smooth affine algebraic group defined over k .

Let $X = \text{Spec}(S)$ be an affine scheme over k , and suppose S is a Krull domain with class group $\text{Cl}(S)$. We assume that k is absolutely algebraically closed in the fraction field L of S ; this is, for example, true when X is an absolutely irreducible variety defined over k . Finally, we assume that G acts on X . Algebraically, then, we have a map $\rho: S \rightarrow S \otimes A$ making the obvious diagrams commute. Let $R = S^G = \{f \in S \mid \rho(f) = f \otimes 1\}$ be the ring of invariant functions.

2. Let T be $(R \setminus \{0\})^{-1}S$, which by [1, p. 10] is again a Krull domain. For $0 \neq f$ in R the element $\rho(f) = f \otimes 1$ is invertible in $T \otimes A$, so ρ extends uniquely to a map $\rho: T \rightarrow T \otimes A$; the appropriate diagrams still commute, since $S \rightarrow T$ is a ring epimorphism. The fraction field K of R is in T and, for f in K , clearly $\rho(f) = f \otimes 1$. If such an f is also in S , it is in R , since $S \otimes A$ injects into $T \otimes A$. Thus $R = S \cap K$.

By [1, p. 7] it follows that R is itself a Krull domain, and that the essential valuations (prime divisors) of R are among the restrictions of the essential valuations of S . Let V_0 be the set of essential valuations of S trivial on R , let

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V_1 be those restricting to essential valuations of R , and V_2 those (if any) which are nontrivial on R but give primes there of height ≥ 2 . The divisor group $\text{Div}(S)$ is the direct sum of corresponding subgroups E_0, E_1, E_2 .

3. LEMMA. *If $0 \neq g$ in L has $\text{div}_S(g)$ in $E_1 + E_2$, then g is in T and is semi-invariant, i.e. $\rho(g) = g \otimes b$ for some character b of G (grouplike element of A).*

PROOF. Each valuation in V_1 and V_2 is nontrivial on an element of $R \setminus \{0\}$, so by [1, p. 10] then T is the intersection of the valuation rings for valuations in V_0 . Thus g and g^{-1} are in T . Hence $\rho(g)$ is an invertible element of $T \otimes A$. But under our hypotheses there is a units theorem for tensor products [5, p. 264] showing that any invertible element in $T \otimes A$ has the form $h \otimes a$ with h and a invertible. Applying the counit ϵ of A we get $g = (\text{id}, \epsilon)\rho(g) = h\epsilon(a)$. Thus $\epsilon(a)$ is nonzero in k . Bringing it across the tensor product we get $\rho(g) = g \otimes b$ with b invertible and $\epsilon(b) = 1$; by [5, p. 265] such b are grouplike. $\square$

4. THEOREM. *Under the hypotheses of (1) there is an abelian group E with two exact sequences*

$$0 \rightarrow E_2 \rightarrow E \rightarrow \text{Cl}(R) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow F \rightarrow E \rightarrow \text{Cl}(S).$$

Here E_2 is the free group generated by essential valuations of S restricting to nonessential nontrivial valuations of R , and F is a quotient of a subgroup of the character group of G .

5. PROOF. For each essential valuation P of R , let $N_P = \sum e_Q Q \in E_1$, where the sum is over the Q restricting to P , and e_Q is the ramification index. (The sum is finite, since only finitely many valuations can be nonzero on a nonzero element in the ideal of P .) Let D be the subgroup of $\text{Div}(S)$ generated by E_2 and the N_P . For $0 \neq f$ in the fraction field K of R we clearly have $\text{div}_S(f) \in D$. Define then

$$E = D / \{ \text{div}_S(f) \mid f \in K^* \}.$$

We can map D onto the divisor group $\text{Div}(R)$ by sending E_2 to zero and N_P to P . For f in K^* the image of $\text{div}_S(f)$ is $\text{div}_R(f)$. The further map onto the class group $\text{Cl}(R)$ thus sends $\text{div}_S(f)$ to zero, and there is an induced homomorphism of E onto $\text{Cl}(R)$. Now if f in K^* has $\text{div}_R(f) = 0$, then f is in $R^* \subseteq S^*$ and $\text{div}_S(f) = 0$; hence E_2 injects into E and, by definition, it goes to zero in $\text{Cl}(R)$. Conversely, if $N = N_2 + \sum m_P N_P$ in D goes to zero in $\text{Cl}(R)$, then $\sum m_P P = \text{div}_R(f)$ for some f in K^* , and $N - \text{div}_S(f)$ is in E_2 . Thus we have an exact sequence $0 \rightarrow E_2 \rightarrow E \rightarrow \text{Cl}(R) \rightarrow 0$. Let $U = \{ g \in L^* \mid \text{div}_S(g) \in D \}$; this is a multiplicative subgroup of L^* , and $\text{div}_S(U) = D \cap \text{div}_S(L^*)$ is the kernel of the obvious map $D \rightarrow \text{Div}(S) \rightarrow \text{Cl}(S)$. As K^* is in U , the map factors to give $E \rightarrow \text{Cl}(S)$. But an element g of U goes to zero in E iff $\text{div}_S(g) = \text{div}_S(f)$ for some f in K^* , which means $g = sf$ for some s in S^* . Thus for $F = U / K^* S^*$ we have $0 \rightarrow F \rightarrow E \rightarrow \text{Cl}(S)$.

By the lemma each element g of U satisfies $\rho(g) = g \otimes b$ for some character b of G . Clearly the map sending g to b is a homomorphism from U to the character group of G , and elements of K^* are mapped to 1. Conversely, suppose $\rho(g) = g \otimes 1$. Since g is in T by the lemma, we can write $g = s/r$ with s in S and r in R . As $\rho(r) = r \otimes 1$, we get $\rho(s) = s \otimes 1$, so s is in R and g is in K . Hence U/K^* injects into the group of characters. But by definition F is a quotient of U/K^* . $\square$

Explicitly we can say that F is given by the characters occurring for semi-invariants in U modulo those occurring for units of S .

6. COROLLARY. *Suppose G is algebraic and $\text{Cl}(S)$ is finitely generated. Then $\text{Cl}(R)$ is finitely generated and V_2 is finite.*

PROOF. When A is a finitely generated algebra, it is well known that the character group is finitely generated [5, p. 266]. $\square$

COROLLARY. *If G has no nontrivial characters, then $\text{Cl}(R)$ is a quotient of a subgroup of $\text{Cl}(S)$. If, in addition, $\text{Cl}(S)$ is torsion, then there is no blowing up, and the map $\text{Cl}(R) \rightarrow \text{Cl}(S)$ is injective.*

PROOF. Obviously $F = 0$ if G has no characters. Then if $\text{Cl}(S)$ is torsion, so is its subgroup E ; hence the free group E_2 is zero, and in the sense of [2, p. 30] there is no blowing up. $\square$

The characters involved are those defined over k , so this corollary applies to anisotropic tori as well as to unipotent and semisimple groups.

7. With further conditions on the blowing up we can get more information.

PROPOSITION. *With the hypotheses of (1), assume also that $R_{Q \cap R}$ is a regular local ring for each Q in V_2 . Then E_2 is a pure subgroup of E , and the torsion subgroup of E maps isomorphically onto the torsion subgroup of $\text{Cl}(R)$.*

PROOF. Suppose N in E_2 is in mE , so that for some N' in D we have $N - mN' = \text{div}_S(f)$ for some f in K^* . Looking in E_1 we see that $\text{div}_R(f)$ is a multiple of m , say mN_0 . Take any Q in V_2 . As $R_{Q \cap R}$ has trivial class group [2, p. 39], there is some g outside Q and some h in K^* such that N_0 agrees with $\text{div}_R(h)$ on R_g . Then f/h^m is a unit in R_g , and hence $\text{ord}_Q(f/h^m) = 0$. Thus $\text{ord}_Q(f)$ is a multiple of m . Hence $\text{div}_S(f)$ and N are multiples of m , and E_2 is pure. The final statement follows formally. $\square$

COROLLARY. *Together with these hypotheses, assume also that every element of S^* is in R^* up to m th powers. Then the m -torsion in $\text{Cl}(R)$ is isomorphic to a subgroup of $\text{Cl}(S)$.*

PROOF. The character group of G is torsion-free [5, p. 266], and hence so is U/K^* . By assumption multiplication by m is surjective on S^*K^*/K^* ; hence $F = U/K^*S^*$ has no m -torsion, and the m -torsion in E injects into $\text{Cl}(S)$. $\square$

8. For k algebraically closed and m prime to the characteristic, the last corollary was proved by Magid [4] using étale fundamental groups. Under the same restrictions he proved a related result on Picard groups, and we conclude by establishing it in general.

PROPOSITION. *With the hypotheses of (1) assume also that $S^* = R^*(S^*)^m$. Then the kernel of $\text{Pic}(R) \rightarrow \text{Pic}(S)$ contains no m -torsion.*

PROOF. Let N be an invertible divisor on R with principal image $\text{div}_S(g)$, and suppose $mN = \text{div}_R(f)$. Then $f = cg^m$ for some c in S^* . By hypothesis we can write $c = c_0c_1^m$; absorbing c_0 into f and c_1 into g , we may assume $f = g^m$. Then g must be in U , and actually $\rho(g) = g \otimes 1$, since $\rho(f) = f \otimes 1$ and the character group has no torsion. Thus g is in K . Since $R = S \cap K$, the inclusion $R \rightarrow S$ induces an injection on invertible divisors, and hence $N = \text{div}_R(g)$. $\square$

In particular, if S is a polynomial ring $k[X_1, \dots, X_n]$, then $\text{Pic}(R)$ is torsion-free.

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