COEFFICIENTS OF INVERSES OF MEROMORPHIC UNIVALENT FUNCTIONS¹

GLENN SCHOBER

ABSTRACT. Sharp bounds are obtained for the coefficients B_{2n-1} , $1 \leq n \leq 7$, of the inverses of normalized meromorphic univalent functions. For the remaining values of n , sharp bounds are given in case initial segments of even coefficients are zero.

1. **Introduction.** Let Σ' denote the family of univalent functions

$$g(z) = z + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

in $\tilde{\Delta} = \{z: 1 < |z| < \infty\}$. If G is the inverse of a function $g \in \Sigma'$ (i.e., $G = g^{-1}$), then G has an expansion

$$G(w) = w + \sum_{n=1}^{\infty} \frac{B_n}{w^n}$$

in some neighborhood of $w = \infty$. It is well known that

$$|B_1| = |b_1| \leq 1 \quad \text{and} \quad |B_2| = |b_2| \leq \frac{2}{3}.$$

G. Springer [8] used variational methods to prove that

$$(1) \quad \left| B_3 + \frac{1}{2} B_1^2 \right| \leq \frac{1}{2} \quad \text{and} \quad |B_3| \leq 1$$

and conjectured that

$$(2) \quad |B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!}, \quad n = 3, 4, 5, \dots$$

Equality in (2) occurs for

$$G(w) = \frac{w}{2} \left[1 + \sqrt{1 - \frac{4\eta}{w^2}} \right] = w - \sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!} \frac{\eta^n}{w^{2n-1}}, \quad |\eta| = 1,$$

which is the inverse function of $g(z) = z + \eta/z$.

Very recently, Y. Kubota [3] has observed (1) by an elementary application of Grunsky's inequalities [2] and has verified (2) for $n = 3, 4$, and 5 by a more elaborate application of Grunsky's inequalities.

In this article we shall give an elementary proof of (2) for $2 \leq n \leq 7$ and for functions whose initial $n-2$ (or $n-3$) even coefficients are zero. Our

Received by the editors March 4, 1977.

AMS (MOS) subject classifications (1970). Primary 30A34.

Key words and phrases. Univalent functions, coefficient estimates.

¹This work was supported in part by grant MPS 75-07123A01 from the National Science Foundation.

© American Mathematical Society 1977

method is a direct application of the “exponentiated” Grunsky inequalities, introduced by C. FitzGerald [1]. In fact, the development parallels FitzGerald’s method for obtaining K. Löwner’s [4] bounds for the coefficients of inverses of functions in the familiar class S .

2. “Exponentiated” Grunsky inequalities. Let $H(\tilde{\Delta})$ be the set of all analytic functions on $\tilde{\Delta}$ with the topology of locally uniform convergence. Then $H(\tilde{\Delta})$ is a locally convex topological vector space, and we denote its dual space by $H'(\tilde{\Delta})$. If $L \in H'(\tilde{\Delta})$ and $h(z, \zeta)$ is analytic in $\tilde{\Delta} \times \tilde{\Delta}$, we define

$$L^2(h(z, \zeta)) = L(L(h(z, \zeta))) \text{ and } |L|^2(h(z, \bar{\zeta})) = L\left(\overline{L(h(z, \bar{\zeta}))}\right),$$

where L is applied first to the function of z and then to the function of ζ . In this framework we state (cf. [2], [6], [7, Theorem 11.3]):

Grunsky’s inequalities. If $g \in \Sigma'$ and $L \in H'(\tilde{\Delta})$, then

$$\left| L^2\left(\log \frac{g(z) - g(\zeta)}{z - \zeta}\right) \right| \leq |L|^2\left(\log \frac{1}{1 - 1/(z\bar{\zeta})}\right).$$

These inequalities may be exponentiated in the following sense. If $\varphi(w) = \sum_{n=0}^{\infty} c_n w^n$ is any entire function and $\varphi^+(w) = \sum_{n=0}^{\infty} |c_n| w^n$, then [1], [6], [7, Theorem 11.16]

$$\left| L^2\left(\varphi \circ \log \frac{g(z) - g(\zeta)}{z - \zeta}\right) \right| \leq |L|^2\left(\varphi^+ \circ \log \frac{1}{1 - 1/(z\bar{\zeta})}\right).$$

In particular, if $\varphi(w) = e^{-w}$, then

$$(3) \quad \left| L^2\left(\frac{z - \zeta}{g(z) - g(\zeta)}\right) \right| \leq |L|^2\left(\frac{1}{1 - 1/(z\bar{\zeta})}\right) = \sum_{n=0}^{\infty} \left| L\left(\frac{1}{z^n}\right) \right|^2.$$

Fix $g \in \Sigma'$, denote $G = g^{-1}$, and define $L \in H'(\tilde{\Delta})$ to be the functional that associates to $h \in H(\tilde{\Delta})$ the coefficient d_N in the expansion of $h \circ G(w) = \sum_{n=-\infty}^{\infty} d_n/w^n$ in a neighborhood of $w = \infty$. If $G(w) = w + \sum_{n=1}^{\infty} B_n/w^n$, then $L^2((z - \zeta)/(g(z) - g(\zeta)))$ is the coefficient β_{NN} in the expansion

$$\frac{G(w) - G(\omega)}{w - \omega} = 1 + \sum_{m,n=1}^{\infty} \frac{\beta_{mn}}{w^m \omega^n}.$$

Since $\beta_{NN} = -B_{2N-1}$, we have the following consequence of (3).

LEMMA. If $g \in \Sigma'$ and $G(w) = g^{-1}(w) = w + \sum_{n=1}^{\infty} B_n/w^n$ in a neighborhood of ∞ , then

$$(4) \quad |B_{2N-1}| \leq 1 + \sum_{n=1}^{N-2} |B_N^{(-n)}|^2 \quad (N = 1, 2, 3, \dots),$$

where

$$[G(w)]^{-n} = \frac{1}{w^n} + \sum_{k=n+2}^{\infty} \frac{B_k^{(-n)}}{w^k}$$

in a neighborhood of ∞ . Of course, the sum in (4) is omitted for $N = 1$ and $N = 2$.

The inequality (4) corresponds directly to the estimate (2.10) of FitzGerald [1]. In the next section we shall explore consequences of this inequality. We shall also make use of the well-known area theorem (cf. [6], [7]).

AREA THEOREM. If $g(z) = z + \sum_{n=1}^{\infty} b_n/z^n$ belongs to Σ' , then $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$. In particular, $|b_1| \leq 1$ with equality iff $g(z) = z + \eta/z$, $|\eta| = 1$.

3. Coefficient results. We shall find several coefficient relations to be useful. If $g(z) = z + \sum_{n=1}^{\infty} b_n/z^n$ and $G(w) = g^{-1}(w) = w + \sum_{n=1}^{\infty} B_n/w^n$ in a neighborhood of ∞ , then

$$\begin{aligned} B_1 &= -b_1, & B_3 &= -b_3 - b_1^2, \\ B_2 &= -b_2, & B_4 &= -b_4 - 3b_1b_2. \end{aligned}$$

In addition, if for $n \geq 1$ we let

$$[G(w)]^{-n} = \frac{1}{w^n} + \sum_{N=n+2}^{\infty} \frac{B_N^{(-n)}}{w^N}$$

in a neighborhood of $w = \infty$, then

$$(5) \quad B_N^{(-n)} = \sum_{\substack{k_1 + \dots + k_m + m = N - n \\ m, k_1, \dots, k_m \geq 1}} \frac{(-1)^m (m+n-1)!}{m! (n-1)!} B_{k_1} B_{k_2} \cdots B_{k_m}.$$

In particular, we shall use

$$\begin{aligned} B_3^{(-1)} &= -B_1, & B_5^{(-3)} &= -3B_1, \\ B_4^{(-1)} &= -B_2, & B_6^{(-3)} &= -3B_2, \\ B_5^{(-1)} &= -B_3 + B_1^2, & B_7^{(-3)} &= -3B_3 + 6B_1^2, \\ B_6^{(-1)} &= -B_4 + 2B_1B_2, & & \\ B_7^{(-1)} &= -B_5 + 2B_1B_3 + B_2^2 - B_1^3, & B_6^{(-4)} &= -4B_1, \\ & & B_7^{(-4)} &= -4B_2, \\ B_4^{(-2)} &= -2B_1, & B_7^{(-5)} &= -5B_1, \\ B_5^{(-2)} &= -2B_2, & & \\ B_6^{(-2)} &= -2B_3 + 3B_1^2, & & \\ B_7^{(-2)} &= -2B_4 + 6B_1B_2, & & \end{aligned}$$

THEOREM 1. If $g \in \Sigma'$ and $G(w) = g^{-1}(w) = w + \sum_{n=1}^{\infty} B_n/w^n$ in a neighborhood of $w = \infty$, then

$$\begin{aligned} |B_1| &\leq 1, & |B_3| &\leq 1, & |B_5| &\leq 2, & |B_7| &\leq 5, \\ |B_9| &\leq 14, & |B_{11}| &\leq 42, & \text{and} & & |B_{13}| &\leq 132. \end{aligned}$$

Equality in any of these occurs only for $g(z) = z + \eta/z$, $|\eta| = 1$.

PROOF. The bounds $|B_1| \leq 1$ and $|B_3| \leq 1$ are special cases of the Lemma for $N = 1$ and $N = 2$. For $N = 3$ we have

$$|B_5| \leq 1 + |B_3^{(-1)}|^2 = 1 + |B_1|^2 \leq 2.$$

If we use the Lemma with $N = 4$, together with the inequalities $2|b_2|^2 \leq 1 - |b_1|^2$ and $|b_1| \leq 1$ from the Area Theorem, we have

$$\begin{aligned} |B_7| &\leq 1 + |B_4^{(-2)}|^2 + |B_4^{(-1)}|^2 = 1 + 4|b_1|^2 + |b_2|^2 \\ &\leq 1 + 4|b_1|^2 + \frac{1}{2}(1 - |b_1|^2) = \frac{3}{2} + \frac{7}{2}|b_1|^2 \leq 5. \end{aligned}$$

With the same tools, for $N = 5$ we obtain

$$\begin{aligned} |B_9| &\leq 1 + |B_5^{(-3)}|^2 + |B_5^{(-2)}|^2 + |B_5^{(-1)}|^2 \\ &= 1 + 9|b_1|^2 + 4|b_2|^2 + |-B_3 + B_1^2|^2 \\ &\leq 1 + 9|b_1|^2 + 2(1 - |b_1|^2) + 4 = 7 + 7|b_1|^2 \leq 14. \end{aligned}$$

Similarly, for $N = 6$ we use the estimates

$$\begin{aligned} |B_6^{(-2)}|^2 &= |-2B_3 + 3B_1^2|^2 \leq 4|B_3|^2 + 12|B_3B_1^2| + 9|B_1|^2 \\ &\leq 4 + 21|B_1|^2 = 4 + 21|b_1|^2 \end{aligned}$$

and

$$|B_6^{(-1)}|^2 = |-B_4 + 2B_1B_2|^2 = |b_4 + 5b_1b_2|^2 \leq 2|b_4|^2 + 50|b_2|^2.$$

Then

$$\begin{aligned} |B_{11}| &\leq 1 + |B_6^{(-4)}|^2 + |B_6^{(-3)}|^2 + |B_6^{(-2)}|^2 + |B_6^{(-1)}|^2 \\ &\leq 1 + 16|b_1|^2 + 9|b_2|^2 + 4 + 21|b_1|^2 + 2|b_4|^2 + 50|b_2|^2 \\ &\leq 5 + 37|b_1|^2 + \frac{59}{2}(1 - |b_1|^2 - 4|b_4|^2) + 2|b_4|^2 \\ &\leq \frac{69}{2} + \frac{15}{2}|b_1|^2 \leq 42. \end{aligned}$$

Finally, for $N = 7$ the estimates

$$\begin{aligned} |B_7^{(-3)}|^2 &= |-3B_3 + 6B_1^2|^2 \leq 9 + 72|B_1|^2 = 9 + 72|b_1|^2, \\ |B_7^{(-2)}|^2 &= |-2B_4 + 6B_1B_2|^2 \leq 8|B_4|^2 + 72|B_2|^2 \\ &= 8|b_4 + 3b_1b_2|^2 + 72|b_2|^2 \leq 32|b_4|^2 + 168|b_2|^2, \end{aligned}$$

and

$$\begin{aligned} |B_7^{(-1)}|^2 &= |-B_5 + 2B_1B_3 + B_2^2 - B_1^3|^2 \leq (4 + |B_2|^2 + |B_1|^3)^2 \\ &\leq 16 + 11|B_2|^2 + 9|B_1|^2 = 16 + 11|b_2|^2 + 9|b_1|^2 \end{aligned}$$

lead to the bound

$$\begin{aligned}
|B_{13}| &\leq 1 + 25|b_1|^2 + 16|b_2|^2 + |B_7^{(-3)}|^2 + |B_7^{(-2)}|^2 + |B_7^{(-1)}|^2 \\
&\leq 26 + 106|b_1|^2 + 195|b_2|^2 + 32|b_4|^2 \\
&\leq 26 + 106|b_1|^2 + \frac{195}{2}(1 - |b_1|^2 - 4|b_4|^2) + 32|b_4|^2 \\
&\leq \frac{247}{2} + \frac{17}{2}|b_1|^2 \leq 132.
\end{aligned}$$

In each case equality occurs only if $|b_1| = 1$, hence, only for $g(z) = z + \eta/z$, $|\eta| = 1$.

Instead of estimating additional coefficients by the same methods, it would seem appropriate now to attempt to prove a general result. So far, we have not been able to establish (2) for all n . However, we shall give a proof of (2) along the lines of FitzGerald [1] in case some of the even coefficients of G vanish.

THEOREM 2. Suppose $g \in \Sigma'$ and $G(w) = g^{-1}(w) = w + \sum_{n=1}^{\infty} B_n/w^n$ in a neighborhood of $w = \infty$. Assume that $B_2 = B_4 = \cdots = B_{N-2} = 0$ if N is even or that $B_2 = B_4 = \cdots = B_{N-3} = 0$ if N is odd. Then

$$|B_{2N-1}| \leq \frac{(2N-2)!}{N!(N-1)!} \quad (N = 1, 2, 3, \dots).$$

Equality for some N occurs only for $g(z) = z + \eta/z$, $|\eta| = 1$.

PROOF. We shall employ an induction argument. By Theorem 1 this theorem is true for $1 \leq N \leq 7$. Now assume the theorem is true for odd subscripts less than $2N-1$.

In order to use the Lemma, we note that each coefficient $B_N^{(-n)}$ for $1 \leq n \leq N-2$ is a polynomial in the first $N-n-1$ coefficients of G . Among these coefficients the even ones are assumed to be zero. Therefore (4) depends only on the odd coefficients of G with subscript at most $N-2 < 2N-1$.

Since the coefficients of

$$(6) \quad G(w) = \frac{w}{2} \left[1 + \sqrt{1 - \frac{4}{w^2}} \right] = w - \sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!} \frac{1}{w^{2n-1}},$$

which is the inverse of $g(z) = z + 1/z$, are negative, the terms in (5) are all nonnegative and as large as possible. Therefore this function G gives rise to the largest possible value for the right-hand side of (4). If, indeed, we have equality in (4) for this function, then this function must be an extremal function, and the theorem follows.

To verify equality in (4) for the function (6), we may simply verify equality in (3) for the functional L used to derive (4). In this case

$$\left| L^2 \left(\frac{z - \zeta}{g(z) - g(\zeta)} \right) \right| = L^2 \left(\frac{1}{1 - 1/(z\zeta)} \right) = \left| \sum_{n=0}^{\infty} L \left(\frac{1}{z^n} \right) \right|^2.$$

Since the coefficients $B_N^{(-n)} = L(1/z^n)$ are real for the function (6), it follows that

$$\left| \sum_{n=0}^{\infty} L\left(\frac{1}{z^n}\right)^2 \right| = \sum_{n=0}^{\infty} \left| L\left(\frac{1}{z^n}\right) \right|^2$$

and that we have equality in (3).

If $g \in \Sigma'$ is an odd function, then $G = g^{-1}$ is also an odd function. Therefore Theorem 2 implies a complete solution for the inverses of odd functions in Σ' .

COROLLARY. *If g is an odd function in Σ' and $G(w) = w + \sum_{n=1}^{\infty} B_{2n-1}/w^{2n-1}$ in a neighborhood of $w = \infty$, then*

$$|B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!}$$

for all $n \geq 1$. Equality for some n occurs only for $g(z) = z + \eta/z$, $|\eta| = 1$.

This corollary was first proved by E. Netanyahu [5]. He obtained it as a consequence of studying the inverses of *nonvanishing* univalent functions

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}$$

in $\tilde{\Delta}$. If $\gamma = g^{-1}$ and

$$\gamma(w) = w + \sum_{n=0}^{\infty} \frac{\beta_n}{w^n}$$

in a neighborhood of $w = \infty$, he used variational methods to obtain the sharp bounds

$$|\beta_n| \leq 2 \frac{(2n)!}{(n+1)!n!}$$

for all n . In this case the extremal functions are $g(z) = z + 2\eta + \eta^2/z$, $|\eta| = 1$.

The simplicity with which the “exponentiated” Grunsky inequalities lead to the coefficient results in the present article is an attractive feature. We believe that they may lead to a solution of the conjecture (2) for all n .

REFERENCES

1. C. FitzGerald, *Quadratic inequalities and coefficient estimates for schlicht functions*, Arch. Rational Mech. Anal. **46** (1972), 356–368.
2. H. Grunsky, *Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen*, Math. Z. **45** (1939), 29–61.
3. Y. Kubota, *Coefficients of meromorphic univalent functions*, Kōdai Math. Sem. Rep. **28** (1977), 253–261.
4. K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*. I, Math. Ann. **89** (1923), 103–121.
5. E. Netanyahu, *Extremal problems for schlicht functions in the exterior of the unit circle*, Canad. J. Math. **17** (1965), 335–341.
6. Chr. Pommerenke, *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
7. G. Schöber, *Univalent functions—selected topics*, Lecture Notes in Math., vol. 478, Springer-Verlag, Berlin and New York, 1975, 200 pp.
8. G. Springer, *The coefficient problem for schlicht mappings of the exterior of the unit circle*, Trans. Amer. Math. Soc. **70** (1951), 421–450.