THE ABSOLUTE SUMMABILITY OF FOURIER SERIES OF A FUNCTION OF WIENER'S CLASS BY NÖRLUND MEANS

S. N. LAL AND SIYA RAM

ABSTRACT. In this paper we prove that under suitable restrictions on $\{p_n\}$ and the integral modulus of continuity of a function f of Wiener's class the Fourier series of f is summable $|N, p_n|$.

1. A 2π -periodic function f is said to be of r -bounded variation in $[0, 2\pi]$ if there exists a positive constant C such that¹

$$\left(\sum_{i=1}^n |f(x_i) - f(x_{i-1})|^r \right)^{1/r} \leq C,$$

for all partitions $0 = x_0 < x_1 < x_2 < \cdots < x_n = 2\pi$. The class of functions of r -bounded variation on $[0, 2\pi]$ introduced by N. Wiener [8] is denoted by V_r . When $r = 1$, V_1 is the ordinary class of functions of bounded variation. For $r > 1$, Wiener's class V_r is a strictly larger class than the class V_1 .

Let $\{p_n\}$ be a sequence of nonnegative numbers and write

$$P_n = \sum_{\nu=0}^n p_\nu, \quad P_{-1} = p_{-1} = 0.$$

The sequence to sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^n p_{n-\nu} S_\nu \quad (P_n > 0)$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{S_n\}$ generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ with the sequence of partial sums S_n is said to be absolutely summable (N, p_n) or summable $|N, p_n|$, if

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty.$$

In case when $p_0 = 1$ and $p_n = 0$ for $n > 0$, the summability $|N, p_n|$ reduces to the absolute convergence of the series $\sum a_n$.

If $f \in L(-\pi, \pi)$, the series

Received by the editors December 7, 1976.

AMS (MOS) subject classifications (1970). Primary 42A28; Secondary 40F05, 40G05.

Key words and phrases. Wiener's class V_r , summable $|N, p_n|$, Parseval's relation, modulus of continuity, integral modulus of continuity.

¹Throughout the paper C denotes a positive constant not necessarily the same at each occurrence.

© American Mathematical Society 1977

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t),$$

where a_n and b_n are given by the usual Euler-Fourier formulae is called the Fourier series of f . We write

$$\phi(t) = f(x+t) + f(x-t) - 2f(x);$$

$$\alpha(t) = \sum_{\nu=0}^{\infty} p_{\nu} \cos \nu t, \quad \beta(t) = \sum_{\nu=0}^{\infty} p_{\nu} \sin \nu t;$$

$$\alpha_n = \int_0^{\pi} \phi(t) \alpha(t) \cos nt \, dt, \quad \beta_n = \int_0^{\pi} \phi(t) \beta(t) \sin nt \, dt.$$

For a function $f(x)$ defined in a closed interval I , the function

$$\omega(\delta) \equiv \omega(\delta; f) = \sup |f(x_1) - f(x_2)|$$

for $x_1, x_2 \in I$ and $|x_1 - x_2| \leq \delta$, is called the modulus of continuity of f . The function

$$\omega_p(\delta) \equiv \omega_p(\delta; f) = \sup_{0 < |h| < \delta} \left(\int_0^{2\pi} |f(x+h) - f(x)|^p dx \right)^{1/p} \quad (p \geq 1)$$

is called the integral modulus of continuity of f . It is well known that $\lim_{p \rightarrow \infty} \omega_p(\delta) = \omega(\delta)$.

2. Regarding the absolute convergence of Fourier series the following result is known [2].

THEOREM A. *Let $1 < p < \infty$, and $1/p + 1/q = 1$. If $f \in V_r$ and*

$$\sum_{n=1}^{\infty} \frac{1}{n^{1-1/2q}} \left(\omega_{r+(2-r)q} \left(\frac{\pi}{n}; f \right) \right)^{1-r/2p} < \infty \quad (1 \leq r < 2p)$$

then the Fourier series of f is absolutely convergent.

If $r \rightarrow \infty$ and $p \rightarrow \infty$ such that $r/p \rightarrow 0$ then the above theorem reduces to the Szász criterion ([7], [1, p. 155]) of absolute convergence. And if $r = 1$ and $p \rightarrow 1$ so that $q \rightarrow \infty$, then the above theorem reduces to the well-known Zygmund's criterion [9, p. 242] of absolute convergence. For absolute Nörlund summability an analogue of Zygmund's result for the class of functions of bounded variation has been recently obtained by Shah [6, Theorem 1] using Karamata's slowly oscillating functions [3]. Shah observed that his theorem supplements an earlier result of one of the authors (see Lal [4]). In this paper we study the absolute Nörlund summability of Fourier series of functions of r -bounded variation and establish the following

THEOREM. *Let $\{p_n\}$ be a nonnegative, nonincreasing sequence such that $\lim_{n \rightarrow \infty} p_n = 0$, $\{p_n - p_{n+1}\}$ is nonincreasing and*

$$(2.1) \quad \sum_{k=n}^{\infty} \frac{p_k^{2p}}{k^2} \leq C \frac{p_n^{2p}}{n} \quad (1 < p < \infty).$$

If $f \in V_r$ and

$$(2.2) \quad \sum_{n=1}^{\infty} \frac{1}{n^{1-1/2q} p_n} (\omega_{r+(2-r)q}(\pi/n))^{1-r/2p} < \infty$$

$$\left(\frac{1}{p} + \frac{1}{q} = 1, 1 \leq r < 2p \right)$$

and

$$(2.3) \quad \sum_{n=1}^{\infty} \frac{\omega(n^{-1})}{n} < \infty,$$

then the Fourier series of f is summable $|N, p_n|$.

3. We need the following lemmas for the proof of our theorem.

LEMMA 1 [5]. If $\{p_n\}$ is nonnegative and nonincreasing, then for $0 \leq a \leq b \leq \infty$, $0 < t \leq \pi$ and any n ,

$$(3.1) \quad \left| \sum_{k=a}^b p_k e^{i(n-k)t} \right| \leq CP(t^{-1}),$$

and

$$(3.2) \quad P(2^\lambda) \leq CP(2^{\lambda-1}),$$

as $\lambda \rightarrow \infty$.

LEMMA 2 [5]. If $\{p_n\}$ is nonnegative and nonincreasing and if we take

$$\gamma(t) = \sum_{\nu=0}^{\infty} p_\nu e^{i\nu t},$$

then for t in (h, π)

$$(3.3) \quad |\gamma(t+2h) - \gamma(t)| \leq Ch t^{-1} P(h^{-1}).$$

LEMMA 3 [5]. If $\{p_n\}$ is nonnegative and nonincreasing and $\{p_n - p_{n+1}\}$ is nonincreasing then

$$n^2(p_n - p_{n+1}) \leq CP_{n-1}.$$

It is easy to prove the following

LEMMA 4. Under the hypothesis (2.1) of the theorem

$$(3.4) \quad \sum_{k=n}^{\infty} \frac{P_k^2}{k^2} \leq C \frac{P_n^2}{n},$$

and

$$(3.5) \quad \sum_{k=n}^{\infty} \frac{P_k}{k^2} \leq C \frac{P_n}{n}.$$

4. Proof of the theorem. We have

$$A_n(x) = \frac{1}{\pi} \int_0^\pi \phi(t) \cos nt \, dt,$$

and therefore

$$\begin{aligned}
 \pi |t_n - t_{n-1}| &= \frac{1}{P_n P_{n-1}} \left| \int_0^\pi \phi(t) \sum_{k=0}^{n-1} (p_k P_n - p_n P_k) \cos(n-k)t \, dt \right| \\
 &\leq \frac{|\alpha_n|}{P_{n-1}} + \frac{|\beta_n|}{P_{n-1}} + \frac{1}{P_{n-1}} \left| \int_0^{1/n} \phi(t) \left(\sum_{k=n}^\infty p_k \cos(n-k)t \right) dt \right| \\
 &\quad + \frac{p_n}{P_n P_{n-1}} \left| \int_0^{1/n} \phi(t) \left(\sum_{k=0}^{n-1} P_k \cos(n-k)t \right) dt \right| \\
 &\quad + \frac{1}{P_{n-1}} \left| \int_{1/n}^\pi \frac{\phi(t)}{2 \sin(t/2)} \left(\sum_{k=n}^\infty (p_k - p_{k+1}) \sin(n-k+\frac{1}{2})t \right) dt \right| \\
 &\quad + \frac{p_n}{P_n P_{n-1}} \left| \int_{1/n}^\pi \frac{\phi(t)}{2 \sin(t/2)} \left(\sum_{k=0}^{n-1} p_k \sin(n-k+\frac{1}{2})t \right) dt \right| \\
 &\quad + \frac{p_n}{2P_{n-1}} \left(1 - \frac{P_{n-1}}{P_n} \right) \left| \int_{1/n}^\pi \phi(t) \, dt \right| = \sum_{i=1}^7 I_i(n), \quad \text{say.}
 \end{aligned}$$

It is clear that in order to establish the theorem we have to show that

$$(4.1) \quad \sum_{n=2}^\infty I_i(n) < \infty \quad (i = 1, 2, \dots, 7).$$

Using the estimates in (3.1) and (3.5) we have

$$\begin{aligned}
 (4.2) \quad \sum_{n=2}^\infty I_3(n) &\leq C \sum_{n=2}^\infty \frac{\omega(n^{-1})}{P_{n-1}} \sum_{k=n}^\infty \frac{P_k}{k^2} \\
 &\leq C \sum_{n=2}^\infty \frac{\omega(n^{-1})}{n} < \infty,
 \end{aligned}$$

by the hypothesis (2.3) of the theorem.

And clearly

$$\begin{aligned}
 (4.3) \quad \sum_{n=2}^\infty I_4(n) &\leq 2 \sum_{n=2}^\infty \frac{p_n}{P_n P_{n-1}} \int_0^{1/n} \omega(t) \left(\sum_{k=0}^{n-1} P_k \right) dt \\
 &\leq 2 \sum_{n=2}^\infty \frac{\omega(n^{-1})}{n} < \infty,
 \end{aligned}$$

by the application of the hypothesis (2.3) of the theorem.

Since $\{p_n - p_{n+1}\}$ is nonincreasing

$$\begin{aligned}
 (4.4) \quad \sum_{n=2}^\infty I_5(n) &\leq C \sum_{n=2}^\infty \frac{(p_n - p_{n+1})}{P_{n-1}} \left(1 + \sum_{k=1}^n \omega\left(\frac{1}{k}\right) \right) \\
 &\leq C + C \sum_{k=1}^\infty \frac{\omega(k^{-1})}{k} < \infty,
 \end{aligned}$$

by the application of Lemma 3 and the hypothesis (2.3) of the theorem.

Applying Lemma 1 we have

$$(4.5) \quad \begin{aligned} \sum_{n=2}^{\infty} I_6(n) &\leq C \sum_{n=2}^{\infty} \frac{p_n}{P_n P_{n-1}} + C \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^n \frac{\omega(k^{-1}) P_k}{k} \\ &\leq C + C \sum_{k=1}^{\infty} \frac{\omega(k^{-1})}{k} < \infty, \end{aligned}$$

under the hypothesis (2.3) of the theorem.

And clearly

$$(4.6) \quad \sum_{n=2}^{\infty} I_7(n) \leq C \sum_{n=2}^{\infty} \frac{p_n^2}{P_n P_{n-1}} < \infty.$$

Under the hypotheses of the theorem $\phi(t)\alpha(t) \in L^2$. Thus α_n/π is the n th Fourier coefficient of an even function belonging to L^2 . Since the Fourier series of $\phi(t+h)\alpha(t+h) - \phi(t-h)\alpha(t-h)$ is

$$- \frac{4}{\pi} \sum_{n=1}^{\infty} \alpha_n \sin nt \sin nh,$$

an application of Parseval's relation gives

$$(4.7) \quad \begin{aligned} \sum_{n=1}^{\infty} \alpha_n^2 \sin^2 nh &\leq C \int_0^{\pi} \{ \phi(t+h)\alpha(t+h) - \phi(t-h)\alpha(t-h) \}^2 dt \\ &\leq C \int_0^{\pi} \alpha^2(t+h) |\phi(t+h) - \phi(t-h)|^2 dt \\ &\quad + C \int_{-h}^h \phi^2(t) \alpha^2(t+2h) dt + C \int_{-h}^h \phi^2(t) \alpha^2(t) dt \\ &\quad + C \int_h^{\pi} \phi^2(t) |\alpha(t+2h) - \alpha(t)|^2 dt \\ &= I_1(h) + I_2(h) + I_3(h) + I_4(h), \quad \text{say.} \end{aligned}$$

Now

$$\begin{aligned} &\sum_{k=1}^{2N} \left[\int_0^{\pi} \alpha^2 \left(u + \frac{k\pi}{N} \right) \left| \phi \left(u + \frac{k\pi}{N} \right) - \phi \left(u + \frac{\overline{k-1}\pi}{N} \right) \right|^2 du \right]^p \\ &\leq \sum_{k=1}^{2N} \left[\int_0^{\pi} \alpha^{2p} \left(u + \frac{k\pi}{N} \right) \left| \phi \left(u + \frac{k\pi}{N} \right) - \phi \left(u + \frac{\overline{k-1}\pi}{N} \right) \right|^r du \right] \\ &\quad \times \left[\int_0^{\pi} \left| \phi \left(u + \frac{k\pi}{N} \right) - \phi \left(u + \frac{\overline{k-1}\pi}{N} \right) \right|^{(2-r)q+r} du \right]^{p/q} \\ &\leq C (\omega_{r+(2-r)q}(\pi/N))^{2p-r} \int_0^{\pi} p^{2p} \left(\frac{1}{u} \right) du, \end{aligned}$$

and therefore

$$\begin{aligned}
 & \int_0^\pi \alpha^2 \left(u + \frac{\pi}{2N} \right) \left[\phi \left(u + \frac{\pi}{2N} \right) - \phi \left(u - \frac{\pi}{2N} \right) \right]^2 du \\
 & \leq CN^{-1/p} \left[\sum_{k=1}^{2N} \left[\int_0^\pi \alpha^2 \left(u + \frac{k\pi}{N} \right) \left[\phi \left(u + \frac{k\pi}{N} \right) - \phi \left(u + \frac{k-1}{N} \pi \right) \right]^2 du \right]^p \right]^{1/p} \\
 & \leq CN^{-1/p} (\omega_{r+(2-r)q}(\pi/N))^{2-r/p} \left[1 + \sum_{k=1}^{\infty} \frac{P_k^{2p}}{k^2} \right]^{1/p}
 \end{aligned}$$

The above estimate combined with hypothesis (2.1) of the theorem gives

$$(4.8) \quad I_1 \left(\frac{\pi}{2N} \right) \leq CN^{-1/p} (\omega_{r+(2-r)q}(\pi/N))^{2-r/p}.$$

By Lemma 1,

$$(4.9) \quad I_2(h) \leq C \int_{-h}^h \omega^2(t) P^2 \left(\frac{1}{t+2h} \right) dt \leq Ch\omega^2(h) P^2(h^{-1}),$$

and

$$(4.10) \quad I_3(h) \leq C\omega^2(h) \int_{1/h}^{\infty} \frac{P^2(t)}{t^2} dt \leq Ch\omega^2(h) P^2(h^{-1}),$$

by the application of the estimate (3.4) of Lemma 4.

Making use of Lemma 2 we have

$$(4.11) \quad I_4(h) \leq Ch^2 P^2 \left(\frac{1}{h} \right) \int_{1/\pi}^{1/h} \omega^2 \left(\frac{1}{t} \right) dt.$$

Putting $h = \pi/2N$ and collecting the estimates in (4.7) through (4.11) we get

$$\begin{aligned}
 \sum_{n=1}^{\infty} \alpha_n^2 \sin^2 \left(\frac{n\pi}{2N} \right) & \leq C \left[N^{-1/p} (\omega_{r+(2-r)q}(\pi/N))^{2-r/p} \right. \\
 & \quad + N^{-1} \omega^2 \left(\frac{\pi}{2N} \right) P^2 \left(\frac{2N}{\pi} \right) \\
 & \quad \left. + N^{-2} P^2 \left(\frac{2N}{\pi} \right) \int_{1/\pi}^{2N/\pi} \omega^2 \left(\frac{1}{t} \right) dt \right].
 \end{aligned}$$

Taking $N = 2^\nu$ and using the above estimate we get

$$\begin{aligned}
 \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} \alpha_n^2 \right)^{1/2} & \leq C \left(\sum_{n=1}^{\infty} \alpha_n^2 \sin^2 \left(\frac{n\pi}{2^{\nu+1}} \right) \right)^{1/2} \\
 & \leq C \left[2^{-\nu/2p} (\omega_{r+(2-r)q}(\pi/2^\nu))^{1-r/2p} \right. \\
 & \quad \left. + 2^{-\nu} P \left(\frac{2^{\nu+1}}{\pi} \right) \left(\int_{1/\pi}^{2^{\nu+1}/\pi} \omega^2 \left(\frac{1}{t} \right) dt \right)^{1/2} \right].
 \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n=2^{r-1}+1}^{2^r} \frac{|\alpha_n|}{P_{n-1}} &\leq \left(\sum_{n=2^{r-1}+1}^{2^r} |\alpha_n|^2 \right)^{1/2} \left(\sum_{n=2^{r-1}+1}^{2^r} P_{n-1}^{-2} \right)^{1/2} \\ &\leq C \left[\frac{2^{r(1-1/p)/2}}{P(2^r)} (\omega_{r+(2-r)q}(\pi/2^r))^{1-r/2p} \right. \\ &\quad \left. + 2^{-r/2} \left(\int_{1/\pi}^{2^{r+1}/\pi} \omega^2 \left(\frac{1}{t} \right) dt \right)^{1/2} \right] \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{n=2}^{\infty} I_1(n) &= \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^{\nu}} \frac{|\alpha_n|}{P_{n-1}} \\ &\leq C \sum_{\nu=1}^{\infty} \frac{2^{\nu/2q}}{P(2^{\nu})} (\omega_{r+(2-r)q}(\pi/2^{\nu}))^{1-r/2p} \\ &\quad + C \sum_{\nu=1}^{\infty} 2^{-\nu/2} \left(\int_{1/\pi}^{2^{\nu+1}/\pi} \omega^2 \left(\frac{1}{t} \right) dt \right)^{1/2} \\ &\leq C \sum_{n=1}^{\infty} \frac{(\omega_{r+(2-r)q}(\pi/n))^{1-r/2p}}{n^{1-1/2q} P_n} + C \sum_{\nu=1}^{\infty} 2^{-\nu/2} \\ &\quad + C \sum_{\nu=1}^{\infty} 2^{-\nu/2} \left(\sum_{k=2}^{2^{\nu}} \omega^2 \left(\frac{1}{k} \right) \right)^{1/2} \\ &\leq C + C \sum_{\nu=1}^{\infty} 2^{-\nu/2} \sum_{m=1}^{\nu} \left(\sum_{n=2^{m-1}+1}^{2^m} \omega^2 \left(\frac{1}{k} \right) \right)^{1/2} \\ &\leq C + C \sum_{m=1}^{\infty} 2^{m/2} \omega \left(\frac{1}{2^m} \right) \sum_{\nu=m}^{\infty} 2^{-\nu/2} \\ &\leq C + C \sum_{m=1}^{\infty} \omega \left(\frac{1}{2^m} \right) < \infty, \end{aligned} \tag{4.12}$$

by the hypotheses (2.2) and (2.3) of the theorem.

Similarly we can show that

$$\sum_{n=2}^{\infty} I_2(n) < \infty. \tag{4.13}$$

Combining the estimates in (4.2) through (4.6) and the estimates in (4.12) and (4.13) the truth of (4.1) follows and the theorem is established.

The authors are thankful to the referee for his kind advice.

REFERENCES

1. N. K. Bary, *A treatise on trigonometric series*. II, Pergamon Press, New York, 1964.
2. M. Izumi and S. Izumi, *On absolute convergence of Fourier series*, Ark. Mat. **7** (1967), 177–184.
3. J. Karamata, *Sur un mode de croissance reguliere*, Bull. Soc. Math. France **61** (1933), 55–62.
4. S. N. Lal, *On a theorem of M. Izumi and S. Izumi*, Pacific J. Math. **35** (1970), 661–667.
5. L. McFadden, *Absolute Nörlund summability*, Duke Math. J. **9** (1942), 168–207.
6. S. M. Shah, *Absolute Nörlund summability of Fourier series*, Tôhoku Math. J. **24** (1970), 309–318.
7. O. Szász, *On the absolute convergence of trigonometric series*, Ann. of Math. **47** (1946), 213–220.
8. N. Wiener, *The quadratic variation of a function and its Fourier coefficients*, Massachusetts J. Math. **3** (1924), 72–94.
9. A. Zygmund, *Trigonometric series*. I, Cambridge Univ. Press, New York, 1959.

DEPARTMENT OF MATHEMATICS, BANARAS HINDU UNIVERSITY, VARANASI-221005, INDIA