SIMILARITY OF CERTAIN OPERATORS IN l^p

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ABSTRACT. Let M be the multiplication operator in l^p , $1 < p < \infty$, i.e., $M: x = \{x_k\} \rightarrow \{kx_k\}$. Let $w = \{w_j\}_{j=0}^\infty$ be a weight, i.e., a positive sequence such that $w_1 < w_0 = 1$ and $w_{n+m} \leq w_n w_m$. For $\zeta \in C$, define N_w^ζ on l^p by

$$(N_w^\zeta x)_k = \sum_{j=1}^k \left(\frac{w_k}{w_j} \right) \binom{\zeta - 1 + k - j}{k - j} x_j \quad (k = 1, 2, \dots).$$

Then $\{N_w^\zeta; \zeta \in C\}$ is a holomorphic group of operators, and for any function g holomorphic on the spectrum of N_w^ζ , $M + g(N_w^\zeta)$ is similar to $M + g(1)I$.

1. Introduction. It was proved in [3] that the operators $M + \lambda J$ and $M + \mu J$ in $L^p(0, 1)$, $1 < p < \infty$ (where $M: f(x) \rightarrow xf(x)$ and $J: f(x) \rightarrow \int_0^x f(t) dt$; $\lambda, \mu \in C$) are similar if and only if $\operatorname{Re} \lambda = \operatorname{Re} \mu$. It is natural to look for an analogous result in l^p , with M the "multiplication operator"

$$(Mx)_k = kx_k \quad (x = \{x_k\}_{k=1}^\infty \in l^p),$$

and with a suitable discrete analog of J . Since M is an *unbounded* (closed, densely defined) operator in l^p , it is convenient to have the corresponding J *bounded*, and the natural choice is the "weighted summation operator" in l^p

$$(1) \quad (N_w x)_k = \sum_{j=1}^k (w_k/w_j) x_j \quad (k = 1, 2, \dots),$$

where the "weight" $w = \{w_j\}_{j=0}^\infty$ is a positive sequence such that

$$(2) \quad w_1 < w_0 = 1,$$

$$(3) \quad w_{n+m} \leq w_n w_m \quad (m, n = 0, 1, 2, \dots).$$

For example, $w_n = \exp(-an^b)$ defines a weight for $a > 0$, $b \geq 1$. On the other hand, any weight w is majorized by an exponential weight $\{e^{-an}\}$, since $w_n \leq w_1^n$ by (3) and $w_1 = e^{-a}$ with $a > 0$ by (2). Note also that w is strictly decreasing ($w_{n+m} \leq w_n w_1^m < w_n$), and $\|w\| = \sum_{n=0}^\infty w_n \leq (1 - w_1)^{-1}$.

The result in [3], and a corresponding result for $L^p(0, \infty)$ [7, p. 1196] (which is a closer analog to our l^p situation) have been imbedded in an abstract theory [4], [7] whose main ingredients are:

(i) J is imbedded in a "regular" semigroup of operators;

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(ii) the pair (M, J) satisfies the "Volterra commutation relation" $[M, J] \subset J^2$ (where $[M, J] = MJ - JM$).

Looking first for an enveloping semigroup for our N_w , we actually discover an enveloping holomorphic group $\{N_w^\zeta; \zeta \in C\}$ given by

$$(4) \quad (N_w^\zeta x)_k = \sum_{j=1}^k \left(\frac{w_k}{w_j} \right) \binom{\zeta - 1 + k - j}{k - j} x_j \quad (x \in l^p),$$

where

$$\binom{\alpha}{n} = \alpha(\alpha - 1) \cdots (\alpha - n + 1)/n! \quad \text{for } n = 1, 2, \dots$$

and

$$\binom{\alpha}{0} = 1 \quad (\alpha \in C).$$

Trivially, $N_w^1 = N_w$, $N_w^0 = I$ (the identity operator), and $(N_w^\zeta x)_1 = x_1$ for all $\zeta \in C$. However, instead of (ii), we have $[M, N_w^\zeta] \subset N_w(N_w^\zeta - I)$, and this "slight" difference leads to totally different results: all the operators $M + \lambda N_w^\zeta$ are similar to $M + \lambda I$; in particular, $M + \lambda N_{w_1}^\zeta$ is similar to $M + \mu N_{w_2}^\omega$ if and only if $\lambda = \mu!$ More generally, our main result (Theorem 2) states that for any weight w , for any $\zeta \in C$, and for any function g holomorphic on the spectrum $\sigma(N_w^\zeta)$ of N_w^ζ , the operator $M + g(N_w^\zeta)$ is similar to $M + g(1)I$. Since $M + \lambda I$ and $M + \mu I$ are similar if and only if $\lambda = \mu$ (by spectrum consideration), our theorem establishes that within the family $\mathfrak{N} = \{M + g(N_w^\zeta)\}$ (where the weight w , the complex number ζ , and the function g holomorphic on $\sigma(N_w^\zeta)$ are parameters), a *complete* set of similarity invariants is $\{g(1)\}$, and $M + g(1)I$ is a canonical similarity model for the class with the invariant $g(1)$. Note that $\mathfrak{N}$ is a family of spectral operators of scalar type (since $M + \lambda I$ is trivially of this type). This should be compared to the negative results for $\text{Re } \zeta = 1$ in L^p (cf. [3], [5, Theorem 12], [7]), and to the positive results for $\text{Re } \zeta > 1$ there (cf. [4, §7] and [7]).

In this paper, we say that a function is holomorphic "on K " (with K compact) if it is holomorphic in a *simply connected* open set containing K .

2. The enveloping group.

THEOREM 1. *For each weight w and $1 \leq p \leq \infty$, $\{N_w^\zeta; \zeta \in C\}$ is a holomorphic group of bounded operators in l^p . Its infinitesimal generator is the bounded operator A_w defined by*

$$(5) \quad (A_w x)_1 = 0, \quad (A_w x)_k = \sum_{j=1}^{k-1} \left(\frac{w_k}{w_j} \right) \frac{x_j}{k - j} \quad (k > 1).$$

In particular,

$$(6) \quad N_w^\zeta = \exp(\zeta A_w), \quad \zeta \in C.$$

PROOF. Since $w_k/w_j \leq w_{k-j}$ ($0 \leq j \leq k$) by (3), and

$$\left| \binom{\zeta - 1 + j}{j} \right| \leq \binom{|\zeta| - 1 + j}{j} \quad (\zeta \in C; j = 0, 1, 2, \dots),$$

we have

$$(7) \quad |(N_w^\zeta x)_k| \leq \sum_{j=1}^k w_{k-j} \binom{|\zeta| - 1 + k - j}{k-j} |x_j| \quad (x \in l^p).$$

Let

$$(k_w^\zeta)_j = \begin{cases} 0, & j < 0, \\ w_j \binom{\zeta - 1 + j}{j}, & j \geq 0, \end{cases}$$

and $(|x|)_k = |x_k|$. Considering l^p as a subspace of $l^p(Z)$, the right-hand side of (7) is the convolution $k_w^{|\zeta|} * |x|$, and it follows that

$$\|N_w^\zeta x\|_p \leq \|k_w^{|\zeta|}\|_1 \|x\|_p.$$

Since

$$\|k_w^{|\zeta|}\|_1 \leq \sum_{j=0}^{\infty} \binom{|\zeta| - 1 + j}{j} w_j = (1 - w_1)^{-|\zeta|},$$

we conclude that for each weight w and $\zeta \in C$,

$$(8) \quad \|N_w^\zeta\| \leq (1 - w_1)^{-|\zeta|}.$$

The group property is verified next. Fix $x \in l^p$, and $\alpha, \beta \in C$. For $n = 1, 2, \dots$,

$$\begin{aligned} (N_w^\alpha N_w^\beta x)_n &= \sum_{k=1}^n \left(\frac{w_n}{w_k} \right) \binom{\alpha - 1 + n - k}{n-k} \sum_{j=1}^k \left(\frac{w_k}{w_j} \right) \binom{\beta - 1 + k - j}{k-j} x_j \\ &= \sum_{j=1}^n \left(\frac{w_n}{w_j} \right) \sum_{r=0}^{n-j} \binom{\alpha - 1 + n - j - r}{n-j-r} \binom{\beta - 1 + r}{r} x_j \\ &= \sum_{j=1}^n \left(\frac{w_n}{w_j} \right) \binom{\alpha + \beta - 1 + n - j}{n-j} x_j \\ &= (N_w^{\alpha+\beta} x)_n. \end{aligned}$$

In the transition from the second to the third line, we used the identity

$$(9) \quad \sum_{r=0}^m \binom{\alpha - 1 + m - r}{m-r} \binom{\beta - 1 + r}{r} = \binom{\alpha + \beta - 1 + m}{m},$$

which follows trivially from the binomial expansion

$$(1 - x)^{-\alpha} = \sum_{j=0}^{\infty} \binom{\alpha - 1 + j}{j} x^j \quad (|x| < 1)$$

and the convolution formula for the coefficients of the product of power series.

Consider now the operator A_w defined in (5). Regarding again l^p as a subspace of $l^p(Z)$, we have

$$|(A_w x)_k| \leq (a_w * |x|)_k$$

for all $x \in l^p$, where $(a_w)_j = 0$ for $j \leq 0$ and $(a_w)_j = w_j/j$ for $j \geq 1$. Hence

$$\|A_w\| \leq \|a_w\|_1 \leq \sum_{j=1}^{\infty} \frac{w_1^j}{j} = -\log(1 - w_1),$$

proving the boundedness of A_w .

For $\zeta \in C \setminus \{0\}$, let $T_w^\zeta = \zeta^{-1}(N_w^\zeta - I) - A_w$. Then for each $x \in l^p$, $(T_w^\zeta x)_k = 0$ for $k = 1, 2$, and

$$(T_w^\zeta x)_k = \sum_{j=1}^{k-2} \left(\frac{w_k}{w_j} \right) \frac{[(\zeta + 1) \cdots (\zeta + k - j - 1) - (k - j - 1)!] x_j}{(k - j)!}$$

for $k > 2$. It follows that

$$|(T_w^\zeta x)_k| \leq (h_w^\zeta * |x|)_k \quad (k = 1, 2, \dots)$$

where $(h_w^\zeta)_k = 0$ for $k \leq 1$ and

$$(h_w^\zeta)_k = w_k |(\zeta + 1) \cdots (\zeta + k - 1) - (k - 1)!| / k! \quad \text{for } k > 1$$

(as usual, l^p is considered as a subspace of $l^p(Z)$). Hence $\|T_w^\zeta\| \leq \|h_w^\zeta\|_1$. For $|\zeta| \leq 1$, the series for $\|h_w^\zeta\|_1$ is majorized by the convergent series $2\sum_{k=2}^{\infty} w_1^k$. Therefore $\|h_w^\zeta\|_1$ is a continuous function of ζ in $|\zeta| \leq 1$, and since $\|h_w^0\|_1 = 0$, it follows that $\|h_w^\zeta\|_1 \rightarrow 0$ as $\zeta \rightarrow 0$, and we conclude that $\lim_{\zeta \rightarrow 0} \zeta^{-1}(N_w^\zeta - I) = A_w$ in the uniform operator topology. Since $\{N_w^\zeta; \zeta \in C\}$ is a group, the proof is complete.

3. The similarity result. We come now to our main result, valid in any l^p ($1 \leq p \leq \infty$) and for any weight w .

THEOREM 2. *For any $\zeta \in C$ and for any complex function g holomorphic on $\sigma(N_w^\zeta)$, $M + g(N_w^\zeta)$ is similar to $M + g(1)I$.*

PROOF. We first reduce the general case to the case $\zeta = 1$.

As in the proof of Theorem 1, one obtains the estimate

$$\|N_w^t\| \leq \sum_{j=0}^{\infty} w_j \left| \binom{it - 1 + j}{j} \right| \quad (t \text{ real}).$$

For $j \geq 1$,

$$\binom{it - 1 + j}{j} = \prod_{m=1}^{j-1} \left(1 + \frac{it}{m} \right) \cdot \frac{it}{j}$$

(where $\prod_1^0 = 1$); hence

$$\begin{aligned} \left| \binom{it - 1 + j}{j} \right|^2 &= \prod_{m=1}^{j-1} \left(1 + \frac{t^2}{m^2} \right) \cdot \left(\frac{t^2}{j^2} \right) \\ &< \prod_{m=1}^{\infty} \left(1 + \frac{t^2}{m^2} \right) = \frac{\sinh \pi t}{\pi t}. \end{aligned}$$

Therefore,

$$(10) \quad \|N_w^{it}\| < \|w\| (\sinh \pi t / \pi t)^{1/2}.$$

Let

$$s^+(A_w) = \sup \operatorname{Im} \sigma(A_w) \quad \text{and} \quad s^-(A_w) = \inf \operatorname{Im} \sigma(A_w).$$

By [2, p. 165],

$$\begin{aligned} s^\pm(A_w) &= \lim_{t \rightarrow \pm \infty} t^{-1} \log \|\exp(-itA_w)\| \\ &= \lim_{t \rightarrow \pm \infty} t^{-1} \log \|N_w^{-it}\|, \end{aligned}$$

and by (10),

$$(11) \quad |s^\pm(A_w)| \leq \frac{1}{2} \lim_{|t| \rightarrow \infty} |t|^{-1} \log |\sinh \pi t| = \pi/2.$$

Since $\sigma(N_w) = \exp \sigma(A_w)$, it follows that

$$|\operatorname{Arg} \sigma(N_w)| = |\operatorname{Im} \sigma(A_w)| \leq \pi/2,$$

i.e., $\sigma(N_w)$ is contained in the closed right half-plane, and since $0 \notin \sigma(N_w)$ (cf. Theorem 1), the nonpositive axis is *not* in $\sigma(N_w)$. Therefore the principal values $\operatorname{Log} \lambda$ and $h_\zeta(\lambda) = \lambda^\zeta = \exp(\zeta \operatorname{Log} \lambda)$ (for any $\zeta \in \mathbb{C}$) are holomorphic on $\sigma(N_w)$. By the composite function theorem [1, p. 171],

$$\operatorname{Log} N_w = \operatorname{Log} \exp(A_w) = A_w$$

and

$$h_\zeta(N_w) = \exp(\zeta \operatorname{Log} N_w) = \exp(\zeta A_w) = N_w^\zeta.$$

Let g be a complex function holomorphic on $\sigma(N_w^\zeta) = h_\zeta(\sigma(N_w))$. Then $f_\zeta = g \circ h_\zeta$ is holomorphic on $\sigma(N_w)$ and $f_\zeta(N_w) = g(N_w^\zeta)$. Assuming Theorem 2 for $\zeta = 1$, we obtain that $M + g(N_w^\zeta) = M + f_\zeta(N_w)$ is similar to $M + f_\zeta(1)I = M + g(1)I$, and the general case is established.

It remains to prove Theorem 2 for $\zeta = 1$. We rely on the following special case of our theorem in [6].

LEMMA. *Let B and C be commuting bounded operators in a Banach space X , and let M be a closed operator in X with B -invariant domain such that $[M, B] \subset C$. Then for any complex function f holomorphic on $\sigma(B)$, $M + f'(B)C$ is similar to M . More precisely,*

$$M + f'(B)C = e^{-f(B)} M e^{f(B)}$$

(as unbounded operators).

In our case, a direct calculation shows that the multiplication operator M in l^p has A_w -invariant domain, and $[M, A_w] \subset N_w - I$. By the lemma,

$$(12) \quad M + f'(A_w)(N_w - I) = \exp[-f(A_w)] M \exp f(A_w)$$

for any complex function f holomorphic on $\sigma(A_w)$. If h is holomorphic on $\sigma(N_w)$, $f(\lambda) = h(e^\lambda)$ is holomorphic on $\sigma(A_w)$, and (12) implies

$$(13) \quad M + h'(N_w)N_w(N_w - I) = \exp[-h(N_w)]M \exp h(N_w).$$

Since $(A_w x)_1 = 0$ for all $x \in l^p$, A_w is not onto, hence $0 \in \sigma(A_w)$, and therefore $1 \in \sigma(N_w)$. Consequently, any g holomorphic on $\sigma(N_w)$ may be written as

$$g(\lambda) = g(1) + (\lambda - 1)g_1(\lambda)$$

with g_1 holomorphic on $\sigma(N_w)$. Since $0 \notin \sigma(N_w)$, $g_1(\lambda)/\lambda$ is holomorphic on $\sigma(N_w)$, and there exists a function h holomorphic on $\sigma(N_w)$ such that $h'(\lambda) = g_1(\lambda)/\lambda$. We have

$$h'(\lambda)\lambda(\lambda - 1) = g(\lambda) - g(1),$$

and therefore (13) implies

$$M + g(N_w) - g(1)I = \exp[-h(N_w)]M \exp h(N_w),$$

i.e.,

$$M + g(N_w) = \exp[-h(N_w)][M + g(1)I]\exp h(N_w).$$

Q.E.D.

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