A VECTOR FIELD WITH ONE ZERO ON G/P

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ABSTRACT. We will give a G_a -action on G/P with only one fixed point, where G is a connected linear algebraic group defined over an algebraically closed field k of characteristic zero and P is a parabolic subgroup of G .

Introduction. When $k = \mathbb{C}$, and X compact Kaehler, the knowledge of such a holomorphic vector field on X gives an elegant way of computing $H^*(X, \mathbb{C})$. We will give an example to show how to use this result.

We will use the following notation: G will be a connected reductive linear algebraic group defined over an algebraically closed field k , B a Borel subgroup of G , T a maximal torus contained in B , Δ the set of simple roots of the set of roots Φ of G with respect to T for the ordering associated with B , and Φ^+ the set of positive roots. Then we have $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$, $\dim \mathfrak{g}_\alpha = 1$, where $\mathfrak{t}$, $\mathfrak{g}$ are the Lie algebras of T and G , respectively. If the characteristic of k is zero, then we have the following.

THEOREM. *There exists a G_a -action on G/B , induced by G , which has exactly one fixed point.*

PROOF. Since the characteristic of k is zero, there is an algebraic group morphism $\varepsilon: G_a \rightarrow G$ such that $d\varepsilon(1) = n = \sum_{\alpha \in S} x_\alpha$, where S is any subset of Φ^+ containing Δ , $0 \neq x_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in S$, $d\varepsilon: k \rightarrow \mathfrak{g}$ is the differential of ε at $0 \in G_a$, and 1 is the unit of k . Now ε induces, via left multiplication, a G_a -action on G/B . We claim $\{[B]\}$ is the only fixed point of this action.

To see this, consider X_n/B in G/B , where $X_n = \{g \in G: \text{Adg}(n) \in b_u = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha\}$. By [1, Chapter 2] X_n/B is a closed subvariety of G/B , and $\{[B]\}$ is the only fixed point of the left action of T on X_n/B . Hence the dimension of X_n/B is zero by [3, p. 153], i.e., $X_n = B$. Now, if gB is a fixed point of the G_a -action ε on G/B , then $g^{-1}\varepsilon(x)g \in B_u$ for all $x \in G_a$, where B_u is the unipotent radical of B . Hence $\text{Adg}^{-1}(d\varepsilon(1)) = \text{Adg}^{-1}(n) \in b_u$. This implies $g^{-1} \in X_n = B$; therefore $\{[B]\}$ is the only fixed point of this action. Q.E.D.

The nilpotent element n , in the form above, was suggested to us by O. Loos.

COROLLARY. *Let G be a connected linear algebraic group defined over an algebraically closed field k of characteristic zero, and let P be a parabolic*

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subgroup of G . Then there exists a G_a -action on G/P , induced by G , having only one fixed point.

PROOF. It is easy to see that we may assume G reductive. In this case, let B be a Borel subgroup contained in P , and let $\varepsilon: G_a \times G/B \rightarrow G/B$ be the given action in the Theorem. Then ε induces a G_a -action on G/P so that the natural morphism $\pi: G/B \rightarrow G/P$, $\pi(gB) = gP$, is G_a -equivariant. By the Borel fixed point theorem and the Theorem above, we conclude that $\{[P]\}$ is the only fixed point of this action. Q.E.D.

Finally we want to give an example to show how to use this Theorem to compute $H^*(\mathbf{P}_n, \mathbb{C})$.

EXAMPLE (CARRELL-LIEBERMAN). Let

$$P = \left\{ \begin{pmatrix} * & & & \\ 0 & & & \\ 0 & * & & \\ \vdots & & & \\ \vdots & & & \\ 0 & & & \end{pmatrix} \right\} \subset \mathrm{GL}_{n+1}(\mathbb{C})$$

be the parabolic subgroup of $\mathrm{GL}_{n+1}(\mathbb{C})$, and $\mathrm{GL}_{n+1}/P = \mathbf{P}_n$, projective n -space.

Take

$$n = \sum_{\alpha \in \Delta} x_{\alpha} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ \vdots & & & & & 1 \\ 0 & 0 & 1 & \cdots & 0 & 0 \end{pmatrix}$$

with ones above the diagonal and zeros elsewhere. Then the associated G_a -action has only one fixed point, $e = [1, 0, 0, \dots, 0]$, by the Corollary. Differentiating this flow gives a holomorphic vector field V with one zero $e = [1, 0, 0, \dots, 0] \in \mathbf{P}_n$. It is easy to see that, in the holomorphic coordinates $z_i = x_{i+1}/x_0$, $i = 1, 2, \dots, n$, where $x_0, \dots, x_n$ are the homogeneous coordinates of $\mathbf{P}_n$, V has the form

$$V|_{(\mathbf{P}_n)_{x_0}} = \sum_{i=1}^n a_i \frac{\partial}{\partial z_i},$$

where $a_i(z_1, \dots, z_n) = z_{i+1} - z_i \cdot z_1$ for $1 \leq i \leq n-1$ and $a_n(z_1, \dots, z_n) = -z_n \cdot z_1$.

Now, we have $\mathbb{C}[z_1, \dots, z_n]/(a_1, \dots, a_n) \cong \mathbb{C}[z_1]/(z_1^{n+1})$, which is the cohomology algebra of $\mathbf{P}_n$ with coefficients in $\mathbb{C}$.

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