$C(X)$ IS NOT NECESSARILY A RETRACT OF 2^X

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ABSTRACT. Let X be a continuum and $2^X(C(X))$ denote the hyperspace of closed subsets (subcontinua) of X . An example is given of a continuum X such that $C(X)$ is not a retract of 2^X .

By a *continuum* X we mean a compact connected metric space. 2^X denotes the hyperspace of closed subsets of X with the finite (Vietoris) topology. $C(X)$ denotes the hyperspace of subcontinua of X with the relative topology from 2^X . Since X is a continuum, each of 2^X and $C(X)$ is also a continuum (see [4]). If $Y \subset X$, then Y is a *retract* of X means that there exists a continuous function $r: X \rightarrow Y$ such that the restriction of r to Y is the identity on Y .

In this paper we obtain a partial answer to a question raised by Sam B. Nadler, Jr. In 1939 Wojdyslawski [7] proved that $C(X)$ is an absolute retract if and only if X is locally connected. Hence $C(X)$ is a retract of 2^X when X is locally connected. In Theorem 3.6 of [5] Nadler shows that $C(X)$ is always a continuous image of 2^X . Following this result, in Question 3.7 of [5], Nadler asks "When is $C(X)$ a retract of 2^X ?" Nadler discusses his question further in [6] and considers the possibility that $C(X)$ is always a retract of 2^X . Using an interesting approach with topological semilattices, Lawson [3] asserted that certain nonlocally connected continua admit a retraction from 2^X onto $C(X)$. However, there is an error in the proof of Proposition 4 in Lawson's paper (his function βi is not a retraction from 2^X onto $C(X)$).¹ Consequently, there is currently no known example of a nonlocally connected continuum X such that $C(X)$ is a retract of 2^X . It is the purpose of this paper to present an example of a continuum X such that $C(X)$ is not a retract of 2^X . The example is a slightly modified version of a subcontinuum of Example 3 of [1].

If $U_1, \dots, U_n$ are open sets in X , then $N(U_1, \dots, U_n) = \{A \in 2^X \mid A \subset \bigcup_{i=1}^n U_i \text{ and for each } i = 1, \dots, n, A \cap U_i \neq \emptyset\}$. The collection of sets $N(U_1, \dots, U_n)$ forms a base for the finite topology on 2^X (see [4]).

EXAMPLE. This example is a subspace of E^3 (Euclidean 3-space). If $x, y \in E^3$ we let $[x, y]$ denote the line segment in E^3 joining x and y . For each

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¹Carl Eberhart and Wiley Williams have observed that the example in this paper admits a semilattice structure which satisfies the hypothesis of Proposition 6 of [3] and, hence, provides a counterexample to Proposition 6.

positive integer n let

$$A_n = [(0, 1/2, 1/n), (0, 1, 0)]$$

and

$$B_n = [(1, 0, 1/n), (1/n, 0, 1/n)] \cup [(1/n, 0, 1/n), (0, 1/2, 1/n)] \\ \cup [(0, 1/2, 1/n), (-1/n, 0, 1/n)] \cup [(-1/n, 0, 1/n), (-1, 0, 1/n)].$$

Let X be the closure of $\bigcup_{n=1}^{\infty} (A_n \cup B_n)$ and let

$$B = \lim_{n \rightarrow \infty} B_n = [(1, 0, 0), (-1, 0, 0)] \cup [(0, 0, 0), (0, 1/2, 0)].$$

Suppose that r is a retraction from 2^X onto $C(X)$. Let $\mathcal{V} = N(V_1, \dots, V_m) \cap C(X)$ be an open set such that $B \in \mathcal{V}$ and such that each V_i has diameter $< 1/2$. Then $(0, 1, 0) \notin \mathcal{V}$. Since $r(B) = B$ and r is continuous at B , there exists an open set $\mathcal{U} = N(U_1, \dots, U_k)$ such that $B \in \mathcal{U} \subset N(V_1, \dots, V_m)$ and $r(\mathcal{U}) \subset \mathcal{V}$. Choose $\delta > 0$ such that, for

$$D = [(1, 0, 0), (-1, 0, 0)] \cup [(0, 0, 0), (0, 1/2 - \delta, 0)],$$

$D \in \mathcal{U}$. Then there exists a sequence $D_n \rightarrow D$ such that for each n , $D_n \subset B_n$ and $D_n \in 2^X - C(X)$. Since $D_n \rightarrow D$, $r(D_n) \rightarrow r(D) = D$. Note that subcontinua of X which are sufficiently close to D must be contained in B . Hence there exists a positive integer n_1 such that for each $n \geq n_1$, $r(D_n) \subset B$. Since $D_n \rightarrow D \in \mathcal{U}$ and $B_n \rightarrow B \in \mathcal{U}$, there is a positive integer n_2 such that for each $n \geq n_2$, $D_n \in \mathcal{U}$ and $B_n \in \mathcal{U}$. It follows easily from the definition of the finite topology that if $n \geq n_2$ and $E \in 2^X$ such that $D_n \subset E \subset B_n$, then $E \in \mathcal{U}$. Let $j = \max\{n_1, n_2\}$. By [2, Lemma 2.3] there is a segment $\sigma: [0, 1] \rightarrow 2^X$ such that $\sigma(0) = D_j$ and $\sigma(1) = B_j$. Since $j \geq n_2$, it follows from [2, Definition 2.2] that $\sigma([0, 1]) \subset \mathcal{U}$. Hence $r(\sigma([0, 1]))$ is a subcontinuum of $C(X)$ contained in $\mathcal{V}$. Furthermore, $r(D_j) \in r(\sigma([0, 1]))$ and $B_j = r(B_j) \in r(\sigma([0, 1]))$. Let $A = \bigcup r(\sigma([0, 1]))$. By [2, Lemma 1.2], A is a subcontinuum of X . Since $B_j \subset A$, $r(D_j) \subset A$, and $r(D_j) \subset B$, it is easy to see that $(0, 1, 0) \in A$. However, since $r(\sigma([0, 1])) \subset \mathcal{V}$, $A \subset \bigcup \mathcal{V}$, and since $(0, 1, 0) \notin \bigcup \mathcal{V}$, $(0, 1, 0) \notin A$. We have a contradiction. Hence $C(X)$ is not a retract of 2^X .

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