ZERO-EXTREME POINTS AND THE GENERALIZED CONVEX KERNEL

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ABSTRACT. Let S be a compact simply connected set in the plane. Let $K(n)$ denote the generalized convex kernel of S of order n , $\text{bd } S$ the boundary of S , $E(0, S)$ the set of 0-extreme points of S , and for $x \in S$, let $K(n, x)$ denote the n th order convex kernel of x in S . It is known that $K(n) = \bigcap \{K(n, x) | x \in \text{bd } S\}$ and in certain cases, it is known that $K(1) = \bigcap \{K(1, x) | x \in E(0, S)\}$. The main result of this paper extends the above mentioned results for certain sets. It is shown that $K(n) = \bigcap \{K(n, x) | x \in E(0, S)\}$ for certain compact simply connected sets S in the plane. In the process of obtaining this result, a characterization of the boundary is also obtained.

1. Introduction. Let S be a compact set in E_2 . A point p in S is called a 0-extreme point of S if and only if it is not interior to a segment in S . The set of 0-extreme points of S will be denoted by $E(0, S)$. A point q in S is called an extreme point of S if and only if it is not interior to a segment with endpoints in S . For x in S , let $K(n, x)$ denote the set of points that can be joined to x by some polygonal k -path in S , where $k \leq n$. The generalized convex kernel $K(n)$ is defined by $K(n) = \{x | S \subset K(n, x)\}$.

In a paper by Kenelly, Hare, Evans and Ludescher [2], a result was proved which in E_2 is as follows: let S be a compact set in E_2 such that $K(1) \neq \emptyset$, then $K(1) = \bigcap \{K(1, x) | x \in E(0, S)\}$.

In a paper by Bruckner and Bruckner [1], the following was proved: let S be a compact simply-connected set in E_2 , then $K(n) = \bigcap \{K(n, x) | x \in \text{bd } S\}$.

The main concern of this paper is to sharpen the second result and to provide a generalization of the first (for certain sets). It will be proved that if S is a compact Jordan set in E_2 , then $K(n) = \bigcap \{K(n, x) | x \in E(0, S)\}$.

There are numerous examples to show that this result cannot be refined further by taking the intersection over the set of extreme points of S . One such example follows.

Example. Let J be the closed polygonal path whose vertices (in order) are as follows: $(-3, -3)$, $(-3, 3)$, $(3, 3)$, $(3, -3)$, $(1, -3)$, $(2, 0)$, $(-2, 0)$ and

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$(-1, -3)$. Then J is a closed Jordan curve. Let S be the set consisting of J along with its interior, then S is a compact simply-connected set. Label the vertices x_1 through x_8 , successively. Then the set of extreme points of S is $\text{Ext}(S) = \{x_1, x_2, x_3, x_4\}$. Let $y = (-3, -1)$, then $y \in \cap \{K(2, x) | x \in \text{Ext}(S)\}$. However, $y \notin K(2)$ since $y \notin K(2, x_5)$.

2. Preliminaries. Throughout this paper, all sets will be in E_2 . If A is a set, then $\bar{A}$ will denote its closure, $\text{bd } A$ its boundary, A^c its complement, and $\text{int } A$ its interior. If B is a set, then $A - B$ will denote the complement of B relative to A . For points p and q , $[p, q]$ will denote the closed segment with endpoints p and q , (p, q) the corresponding open segment, and $L(p, q)$ the line determined by p and q .

DEFINITION. A compact set S is said to be *simply-connected* if and only if S^c is connected.

It is most important to keep in mind that if J is a closed Jordan curve in the simply-connected set S , then the interior of J is contained in S , where the interior of J is in the sense of the Jordan Curve Theorem.

DEFINITION. The set S is said to be an L_n set if $K(n, x) = S$ for every $x \in S$; i.e., $K(n) = S$.

3. Properties of the boundary.

DEFINITION. A *domain* is an open connected set. A domain whose boundary is a simple closed curve is called a *Jordan domain*. A compact set which is the closure of a Jordan domain is called a *compact Jordan set*.

DEFINITION. Let D be a domain. A *cross-cut* of D is a simple arc whose endpoints are in $\text{bd } D$ and all its other points are in D .

THEOREM 1. Let S be a compact Jordan set with boundary J . If z is not a point of local convexity of S^c , then $z \in \overline{E(0, S)}$.

PROOF. Let S be a compact Jordan set with boundary J and interior D . Suppose $z \in J$ and z is not a point of local convexity of S^c .

Let $\epsilon > 0$ and let ϵ^* satisfy the inequality $0 < \epsilon^* < \epsilon$. Then the ϵ^* -neighborhood of z , $N(\epsilon^*; z)$, has the property that $N(\epsilon^*; z) \cap S^c$ is not convex. Thus, there exist $p_1, p_2 \in N(\epsilon^*; z) \cap S^c$ such that $[p_1, p_2] \not\subset N(\epsilon^*; z) \cap S^c$. Hence, it follows that $[p_1, p_2] \not\subset S^c$ and thus $[p_1, p_2] \cap S \neq \emptyset$. Since J is a Jordan curve and S^c is open, we know that p_1 and p_2 can be chosen to have the additional properties that $[p_1, p_2]$ contains a cross-cut $[q_1, q_2]$ of D and that one of the two components of $D - [q_1, q_2]$ is a subset of $N(\epsilon^*; z)$. Denote this component by D_1 and let L_1 be the subarc of J such that $\text{bd } D_1 = [q_1, q_2] \cup L_1$. (See Figures 1 and 2.)

Let $L(q_1, q_2)$ denote the line determined by q_1 and q_2 . Now D_1 is an open, nonempty set and $\overline{D_1}$ is compact. Hence, there exists a line H such that:

1. H is parallel to $L(q_1, q_2)$ but $H \neq L(q_1, q_2)$;
2. $H \cap \overline{L_1} \neq \emptyset$;
3. $H \cap \overline{D_1} = H \cap L_1$.

FIGURE 1

FIGURE 2

Since $\overline{D_1}$ is compact and H is closed, it follows that $H \cap \overline{D_1}$ is compact. Hence, $H \cap L_1$ is compact. Since $H \cap L_1$ is a nonempty compact subset of the line H , its components consist of closed segments and/or singleton sets. If $\{y\}$ is a component of $H \cap L_1$, then clearly $y \in E(0, S)$. If $[y_1, y_2]$ is a component of $H \cap L_1$, then clearly $y_1, y_2 \in E(0, S)$. Thus, $(H \cap L_1) \cap E(0, S) \neq \emptyset$.

Now, $D_1 \subset N(\epsilon^*; z)$ implies that $\overline{D_1} \subset N(\epsilon; z)$. Thus $(H \cap \overline{D_1}) \subset N(\epsilon; z)$ and hence $(H \cap L_1) \subset N(\epsilon; z)$. It now follows that $N(\epsilon; z) \cap E(0, S) \neq \emptyset$. Since $\epsilon > 0$ was arbitrary, it follows that $z \in \overline{E(0, S)}$.

DEFINITION. Let S be a compact Jordan set. A Jordan arc L is said to be a *concave arc of S* if:

- (i) $L \subset \text{bd } S$,

- (ii) the endpoints of L are in $\overline{E(0, S)}$, and
- (iii) if $x \in L$ and x is not an endpoint of L , then x is a point of local convexity of S^c .

THEOREM 2. *Let S be a compact Jordan set and let $\Phi(S) = \overline{E(0, S)}^c \cap \text{bd } S$. Let L be an arc with endpoints a and b . Then L is a concave arc of S if and only if $L - \{a, b\}$ is a component of $\Phi(S)$.*

PROOF. Suppose that $L - \{a, b\}$ is a component of $\Phi(S)$ and let $x \in L - \{a, b\}$.

If x is not a point of local convexity of S^c , then $x \in \overline{E(0, S)}$ by Theorem 1. However, this is impossible by the definition of $\Phi(S)$. Thus it follows that x is a point of local convexity of S^c . Hence, L is a concave arc of S .

Now suppose that L is a concave arc of S .

If $x \in L - \{a, b\}$, it is easily seen that $x \notin E(0, S)$. It follows that $L - \{a, b\}$ is a component of $\Phi(S)$.

THEOREM 3. *Let S be a compact Jordan set and let L be a concave arc of S with endpoints a and b . If $x \in S$ is such that $a, b \in K(n, x)$, then $L \subset K(n, x)$.*

PROOF. Suppose $P_n(x, a)$ and $P_n(x, b)$ are n -paths in S joining x to a and b respectively. Letting $P_n = P_n(x, a) \cup P_n(x, b)$, it is clear that $P_n \subset K(n, x)$.

Let $y \in L - \{a, b\}$, then $y \in \text{bd } S$. Also, from Theorem 2, we know that $y \notin E(0, S)$. Thus, there is a segment $[p, q]$ such that $[p, q] \subset S$ and $y \in (p, q)$. By the compactness of S , $[p, q]$ can be extended to a maximal segment $[p^*, q^*]$ in S such that $[p, q] \subset [p^*, q^*]$. Since L is a concave arc and S is simply-connected, it follows that $P_n \cap [p^*, y] \neq \emptyset$ and $P_n \cap [y, q^*] \neq \emptyset$ (see Figure 3). (Recall that since S is simply-connected, the points "enclosed

FIGURE 3

by $P_n \cup L$ " are also in S . Thus, if either $P_n \cap [p^*, y]$ or $P_n \cap [y, q^*]$ were empty, then $[p^*, q^*]$ could be extended, thus contradicting its maximality.) Letting $p_1 \in P_n \cap [p^*, y]$ and $q_1 \in P_n \cap [y, q^*]$, it follows that $p_1, q_1 \in K(n, x)$ and $y \in [p_1, q_1]$. From [4], we know that $[p_1, q_1] \subset K(n, x)$ and thus $y \in K(n, x)$. Since y was arbitrary in $L - \{a, b\}$, it is clear that $L - \{a, b\} \subset K(n, x)$. Since $a, b \in K(n, x)$, it now follows that $L \subset K(n, x)$.

THEOREM 4. Let S be a compact Jordan set. Then $K(n) = \cap \{K(n, x) | x \in E(0, S)\}$.

PROOF. Since $E(0, S) \subset \text{bd } S$, it follows that

$$\cap \{K(n, x) | x \in \text{bd } S\} \subset \cap \{K(n, x) | x \in E(0, S)\}.$$

From [1], we know that $K(n) = \cap \{K(n, x) | x \in \text{bd } S\}$. Thus, it is clear that $K(n) \subset \cap \{K(n, x) | x \in E(0, S)\}$. To prove the desired equality, we need only show the reverse containment.

Thus, suppose $z \in \cap \{K(n, x) | x \in E(0, S)\}$. Then $E(0, S) \subset K(n, z)$. From [1], we know that $K(n, z)$ is compact. Hence, $\overline{E(0, S)} \subset K(n, z)$. By Theorems 2 and 3, it follows that $\Phi(S) \subset K(n, z)$. Since $\text{bd } S = \Phi(S) \cup \overline{E(0, S)}$, it is clear that $\text{bd } S \subset K(n, z)$. Hence, $z \in K(n, y)$ for every $y \in \text{bd } S$ and therefore $z \in \cap \{K(n, y) | y \in \text{bd } S\}$. Thus, $z \in K(n)$. Hence, the reverse containment has been verified and thus the desired equality has been proved.

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