SUM-FREE SETS OF INTEGERS

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ABSTRACT. A set S of integers is said to be sum-free if $a, b \in S$ implies $a + b \notin S$. In this paper, we investigate two new problems on sum-free sets: (1) Let $f(k)$ denote the largest positive integer for which there exists a partition of $\{1, 2, \dots, f(k)\}$ into k sum-free sets, and let $h(k)$ denote the largest positive integer for which there exists a partition of $\{1, 2, \dots, h(k)\}$ into k sets which are sum-free mod $h(k) + 1$. We obtain evidence to support the conjecture that $f(k) = h(k)$ for all k . (2) Let $g(n, k)$ denote the cardinality of a largest subset of $\{1, 2, \dots, n\}$ that can be partitioned into k sum-free sets. We obtain upper and lower bounds for $g(n, k)$. We also show that $g(n, 1) = \lfloor (n+1)/2 \rfloor$ and indicate how one may show that for all $n < 54$, $g(n, 2) = n - \lfloor n/5 \rfloor$.

A set S of integers is said to be sum-free if $a, b \in S$ implies $a + b \notin S$. The case $a = b$ is not excluded, so that $a \in S$ implies $2a \notin S$. A well-known theorem of I. Schur [14] states that if the set $\{1, 2, \dots, [k!e]\}$ is partitioned arbitrarily into k sets, at least one of the sets fails to be sum-free. Thus we may define $f(k)$ to be the largest positive integer for which there exists some way of partitioning $\{1, 2, \dots, f(k)\}$ into k sum-free sets.

The determination of the numbers $f(k)$ is a notoriously difficult problem. It is easy to verify that $f(1) = 1$, $f(2) = 4$ and with a little effort one can show that $f(3) = 13$. L. D. Baumert [4], with the aid of a computer, showed that $f(4) = 44$. The value of $f(5)$ seems to be out of reach at the present time. Recently H. Fredericksen [9] proved that $f(5) \geq 138$ and this seems to be the record.

Schur proved that $f(k+1) \geq 3f(k) + 1$ and this, together with his theorem mentioned in the first paragraph, shows that

$$(1) \quad (3^k - 1)/2 \leq f(k) \leq [k!e] - 1.$$

Abbott and Hanson [3], improving on an earlier inequality of Abbott and Moser [2], proved that for all positive integers k and l ,

$$(2) \quad f(k+l) \geq 2f(k)f(l) + f(k) + f(l).$$

From (2), or the earlier inequality of Abbott and Moser, it follows, via a well-known argument, that $\alpha = \lim_{k \rightarrow \infty} f(k)^{1/k}$ exists, although α may be

Received by the editors December 20, 1976 and, in revised form, April 29, 1977.

AMS (MOS) subject classifications (1970). Primary 10L10.

Key words and phrases. Sum-free set, k -wise sum-free set.

¹Supported by the National Research Council of Canada under grant NRC A-2971.

²Supported by the National Research Council of Canada under grant NRC A-9121 and by a Summer Research Fellowship from Wilfrid Laurier University.

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infinite. Schur's lower bound shows that $\alpha \geq 3$. The result of Baumert, with (2), gives $\alpha \geq (89)^{1/4} = 3.0714 \dots$, while Frederickson's lower bound for $f(5)$ gives $\alpha \geq (277)^{1/5} = 3.0796 \dots$. The upper bound given in (1) has been improved slightly by R. W. Irving [11] to $f(k) \leq [k!(e - \frac{1}{24})]$ by appealing to a result of E. G. Whitehead [16] on Ramsey numbers. An account of most of these results may be found in the article by A. P. Street [15]. See also the interesting paper of L. Mirsky [13].

In this paper we investigate two new problems on sum-free sets.

Problem I. Denote by $h(k)$ the largest integer m for which there exists some way of partitioning $\{1, 2, \dots, m\}$ into k sets which are sum-free modulo $m+1$; that is, which contain no solution of $x + y \equiv z \pmod{m+1}$. The partitions which establish the values $f(1) = 1, f(2) = 4, f(3) = 13, f(4) = 44$ are sum-free modulo 2, 5, 14, 45, respectively, and thus $h(k) = f(k)$ for $k \leq 4$. This suggests the conjecture $h(k) = f(k)$ for all k . We have not been able to prove this conjecture but we obtain some evidence to support it by showing that h satisfies the same recurrence inequality (2) as f .

Problem II. We say that a set of integers is k -wise sum-free (or k -SF) if it can be partitioned into k sum-free sets. Let n be a positive integer and let $N_n = \{1, 2, \dots, n\}$. Denote by $g(n, k)$ the size of a largest k -SF subset of N_n . It is not hard to evaluate $g(n, 1)$. In this paper we indicate how one may evaluate $g(n, 2)$ for $n \leq 54$ and obtain some general upper and lower bounds for $g(n, k)$.

THEOREM 1. *For all positive integers k and l ,*

$$h(k+l) \geq 2h(k)h(l) + h(k) + h(l).$$

PROOF. The argument parallels closely the proof of (2) as given in [1]. Partition $\{1, 2, \dots, h(k)\}$ into sets $A_1, A_2, \dots, A_k$ which are SF mod $(h(k) + 1)$ and partition $\{1, 2, \dots, h(l)\}$ into sets $B_1, B_2, \dots, B_l$ which are SF mod $(h(l) + 1)$. For $1 \leq i \leq k$ let

$$C_i = \{s(2h(k) + 1) + t : s = 0, 1, \dots, h(l), t \in A_i\},$$

and for $k < i \leq k + l$ let

$$C_i = \{s(2h(k) + 1) - t : s \in B_{i-k}, t = 0, 1, \dots, h(k)\}.$$

Then it is easy to check that $C_i \cap C_j = \emptyset$ if $i \neq j$ and that $\bigcup_{i=1}^{k+l} C_i = \{1, 2, \dots, m-1\}$ where $m = 2h(k)h(l) + h(k) + h(l) + 1$. We claim that C_i is SF mod m .

Suppose that for some i , $1 \leq i \leq k$, there are numbers $a, b, c \in C_i$ such that

$$(3) \quad a + b \equiv c \pmod{m}.$$

We have

$$(4) \quad \begin{aligned} a &= s_a(2h(k) + 1) + t_a, & b &= s_b(2h(k) + 1) + t_b, \\ c &= s_c(2h(k) + 1) + t_c \end{aligned}$$

where $0 \leq s_a, s_b, s_c \leq h(l)$ and $t_a, t_b, t_c \in A_i$. It is easy to check that $-m + 3 \leq a + b - c \leq 2m - 3$ and, hence, in order for (3) to hold, we must have

$$(5) \quad a + b - c = rm \quad \text{where } r = 0 \text{ or } 1.$$

It follows from (4) and (5) that

$$(6) \quad (s_a + s_b - s_c - rh(l))(2h(k) + 1) = r(h(k) + 1) - t_a - t_b + t_c.$$

The right side of (6) lies between $-2h(k) + 1$ and $2h(k) - 1$ and must therefore be zero. It follows that $t_a + t_b = t_c + r(h(k) + 1)$, and this contradicts the fact that A_i is SF (mod $(h(k) + 1)$).

A similar contradiction arises in case $k < i \leq k + l$, the only difference being that one gets, instead of (6),

$$(7) \quad (s_a + s_b - s_c - r(h(l) + 1))(2h(k) + 1) = t_a + t_b - t_c - rh(k).$$

The right side of (7) lies between $-2h(k)$ and $2h(k)$ and must therefore be zero. Thus $s_a + s_b = s_c = r(h(l) + 1)$, contradicting the fact that B_{i-k} is SF (mod $h(l) + 1$). This completes the proof.

COROLLARY 1. $\beta = \lim_{k \rightarrow \infty} h(k)^{1/k}$ exists.

REMARK 1. The corollary follows easily from Theorem 1 via a well-known argument which we do not give here (see, for example, [2]). It would be of interest to know whether $\alpha = \beta$. We point out that the partition given by Fredericksen to show that $f(5) \geq 138$ is SF (mod 139). Thus $\beta \geq (277)^{1/5}$.

We now discuss Problem II. It is clear that $g(n, k)$ is a nondecreasing function of k , and that if $n \leq f(k)$, then $g(n, k) = n$. It is not hard to evaluate $g(n, 1)$. We state this as a theorem.

THEOREM 2. $g(n, 1) = \lfloor (n + 1)/2 \rfloor$.

PROOF. Let $A = \{a_1, a_2, \dots, a_t\} \subseteq N_n$ be a SF set, $1 \leq a_1 < a_2 < \dots < a_t$. Then the $2t - 1$ numbers $a_1, a_2, \dots, a_t, a_t - a_1, a_t - a_2, \dots, a_t - a_{t-1}$ are distinct and belong to N_n . Thus $2t - 1 \leq n$ so that $g(n, 1) \leq \lfloor (n + 1)/2 \rfloor$. Since the set O_n of odd integers in N_n is SF and has $\lfloor (n + 1)/2 \rfloor$ elements, we have $g(n, 1) = \lfloor (n + 1)/2 \rfloor$.

REMARK 2. O_n is not the unique largest SF subset of N_n , since if $n = 2t$, the set $\{t + 1, t + 2, \dots, 2t\}$ is SF and if $n = 2t + 1$, the set $\{t + 1, t + 2, \dots, 2t + 1\}$ is SF.

REMARK 3. It is interesting to note that if we consider any set of n integers, not necessarily N_n , then it may not be possible to select a SF subset with $\lfloor (n + 1)/2 \rfloor$ elements, e.g., consider the set $\{2, 3, 4, 5, 6, 8, 10\}$. (This example is due to D. Klärner and is quoted in [8].) In fact, if one denotes by $\phi(n)$ the largest integer such that every set of n real numbers, different from zero, always contains a SF subset with $\phi(n)$ elements, then it is known [8] that $n/3 \leq \phi(n) \leq 3n/7$. Other bounds for $\phi(n)$ have been obtained by Erdős, Choi, Komlós, Sulyok and Szemerédi [5]–[8], [12]. Erdős also mentioned that $\phi(n) = \lfloor (n + 2)/2 \rfloor$ is likely if we exclude sums of equal summands.

THEOREM 3. $g(n, k) \geq n - [n/(h(k) + 1)]$.

PROOF. Partition $\{1, 2, \dots, h(k)\}$ into sets $A_1, A_2, \dots, A_k$ which are SF (mod $h(k) + 1$) and let

$$B_i = \{b: 1 \leq b \leq n, b \equiv a \pmod{h(k) + 1} \text{ for some } a \in A_i\}.$$

Then it is clear that B_i is SF and the theorem follows.

COROLLARY 2. $g(n, 2) \geq n - [n/5]$, $g(n, 3) \geq n - [n/14]$, $g(n, 4) \geq n - [n/45]$.

We conjecture that equality holds in Theorem 3, but we have not been able to prove this, even in the case $k = 2$. We now prove a lemma which enables us to show that $g(n, 2) = n - [n/5]$ for $n \leq 54$.

LEMMA 1. For positive integers a and b such that $b + 4a \leq n$, let $P_{a,b} = \{b, b + a, b + 2a, b + 3a, b + 4a\}$ and $Q_{a,b} = \{a, 2a\} \cup P_{a,b}$. For positive integers a and b such that $a < b$ and $2b + a \leq n$, let $R_{a,b} = \{a, b, 2b, b \pm a, 2b \pm a\}$. Let $N_n = A \cup B \cup C$ be a partition where A and B are SF. Then

- (i) $C \cap Q_{a,b} \neq \emptyset$;
- (ii) if $a \notin C$ and $2a \notin C$ then $C \cap P_{a,b} \neq \emptyset$;
- (iii) $C \cap R_{a,b} \neq \emptyset$.

PROOF. (i) Suppose $C \cap Q_{a,b} = \emptyset$. Then $a \notin C$ and we may suppose without loss of generality that $a \in A$. Then $2a \in B$. If $b \in B$ then $b + 2a \in A$ and $b + 3a \in B$. Thus $b + a \notin A \cup B$, so that $b + a \in C$, a contradiction. Thus $b \in A$. But then $b + a \in B$, $b + 3a \in A$ and $b + 4a \in B$, so that $b + 2a \notin A \cup B$. Thus $b + 2a \in C$, a contradiction. This proves (i).

(ii) Follows immediately from (i).

(iii) Suppose $C \cap R_{a,b} = \emptyset$. Let $a \in A$, say. If $b \in A$, we have $b \pm a \in B$ so that $2b \notin A \cup B$, a contradiction. Thus $b \in B$. Then $2b \in A$, $2b \pm a \in B$ and, hence, $b \pm a \in A$, which again gives a contradiction. This completes the proof of the lemma.

The lemma, simple though it is, enables us to narrow considerably the number of possibilities for C . To illustrate, we evaluate $g(20, 2)$.

THEOREM 4. $g(20, 2) = 16$.

PROOF. Suppose not. Then there exists a partition $N_{20} = A \cup B \cup C$ where A and B are SF and $|C| = 3$. By Lemma 1(i), $C \cap Q_{3,3} \neq \emptyset$ and $C \cap Q_{4,4} \neq \emptyset$ and, since $Q_{3,3} \cap Q_{4,4} = \{12\}$, C cannot contain both 1 and 2. For otherwise, $C = \{1, 2, 12\}$ and hence $C \cap R_{3,7} = \emptyset$, contradicting (iii). Hence there are three cases to be considered. We arrive at a contradiction in each case.

Case 1. $1 \notin C$, $2 \notin C$. Then C meets each of $P_{1,6}$, $P_{1,11}$ and $P_{1,16}$. Since these three sets are pairwise disjoint, C must be contained in their union. Hence $C \cap Q_{1,1} = \emptyset$, contradicting (i).

Case 2. $1 \in C$, $2 \notin C$. If $4 \notin C$, then C meets each of the sets $P_{2,2}$, $P_{2,3}$ and

$P_{2,12}$. Since these three sets are pairwise disjoint and none of them contains 1, we get $|C| \geq 4$, a contradiction. Hence $4 \in C$. If $3 \notin C$ and $6 \notin C$ then C meets $P_{3,2}$ and $P_{3,3}$. Since these two sets are disjoint and neither contains 1 or 4, we get $|C| \geq 4$, a contradiction. Thus $C = \{1, 3, 4\}$ or $\{1, 4, 6\}$. In either case $C \cap R_{2,7} = \emptyset$, which contradicts (iii).

Case 3. $1 \notin C$, $2 \in C$. If $3 \notin C$ and $6 \notin C$ then C meets each of the sets $P_{3,1}$, $P_{3,3}$ and $P_{3,5}$. Since these sets are pairwise disjoint and none of them contains 2, we get $|C| \geq 4$, a contradiction. Thus $3 \in C$ or $6 \in C$. If $4 \notin C$ and $8 \notin C$, then since C meets each of the sets $P_{4,1}$, $P_{4,3}$ and $P_{4,4}$, we again get a contradiction. Thus $4 \in C$ or $8 \in C$. Thus C is one of the sets $\{2, 3, 4\}$, $\{2, 3, 8\}$, $\{2, 4, 6\}$, $\{2, 6, 8\}$. But $\{2, 3, 4\} \cap R_{1,6} = \{2, 3, 8\} \cap R_{1,5} = \{2, 4, 6\} \cap R_{1,8} = \{2, 6, 8\} \cap R_{3,4} = \emptyset$, contradicting Lemma 1(iii). This completes the proof.

REMARK 4. Using the same method we have verified that $g(n, 2) = n - [n/5]$ for $n \leq 54$. For the larger values of n , however, the number of possibilities for C becomes greater and the case-by-case argument becomes very tedious. It is possible that by using Baumert's algorithm [4], one may be able to evaluate $g(n, 2)$ for a few more values of n , but since it is unlikely that any insight into what happens for general n will result, we have not looked into this.

We now obtain a general upper bound for $g(n, k)$. We write $m = f(k) + 1$, and let

$$\alpha_k = \liminf_{k \rightarrow \infty} \frac{g(n, k)}{n} \leq \limsup_{n \rightarrow \infty} \frac{g(n, k)}{n} = \beta_k.$$

It is not hard to show that $\beta_k \leq 1 - 1/m^2$. We now indicate how this may be improved.

THEOREM 5.

$$\beta_k \leq 1 - \frac{1}{m} \prod_{p \leq m} \left(1 - \frac{1}{p}\right) < 1 - \frac{c}{m \log m},$$

where the product is taken over all primes $p \leq m$, and where c is a positive constant.

PROOF. Let $N_n = A_1 \cup A_2 \cup \dots \cup A_k \cup C$ be a partition in which each A_i is SF. For each a such that $ma \leq n$ let $Q_a = \{a, 2a, \dots, ma\}$. It is clear that C has nonempty intersection with each Q_a , since otherwise we would get a partition of $\{1, 2, \dots, m\}$ into k sum-free sets. We now show that there is a large collection of Q 's which are pairwise disjoint. If $a, b \leq n/m$, where $a \neq b$ and all prime factors of a and b exceed m , then $Q_a \cap Q_b = \emptyset$. Thus

$$|C| \geq |\{a : a \leq n/m, \text{ all prime factors of } a \text{ exceed } m\}|.$$

Let $P = \prod_{p \leq m} p$ and let μ denote the usual Möbius function (see [10, p. 234]). Then, by the inclusion-exclusion principle (see [10, p. 234, Theorem 261]), and keeping in mind that all prime factors of 1 exceed m , we get

$$\begin{aligned}
|C| &\geq \sum_{d|P} \mu(d) \left[\frac{n}{md} \right] \geq \frac{n}{m} \sum_{d|P} \frac{\mu(d)}{d} - \sum_{d|P} 1 \\
&= \frac{n}{m} \prod_{p \leq m} \left(1 - \frac{1}{p} \right) - 2^{\pi(m)},
\end{aligned}$$

where $\pi(m)$ is the number of primes not exceeding m . It follows that

$$\beta_k \leq 1 - \frac{1}{m} \prod_{p \leq m} \left(1 - \frac{1}{p} \right) < 1 - \frac{c}{m \log m},$$

where the last inequality follows from the theorem of Mertens (see [10, p. 351]).

REMARK 5. It would be of interest to know whether Theorem 5 can be sharpened to $\beta_k \leq 1 - c/m$, for some positive constant c . With regard to the problem of getting a lower bound for α_k , we point out that we have not been able to do better than Theorem 3. If our conjecture that $h(k) = f(k)$ is true then, of course, $\alpha_k \geq 1 - 1/m$. It would be of interest to prove an inequality of the form $\alpha_k \geq 1 - c/m$ without appealing to any unproved conjecture, but we have not been able to do this.

REFERENCES

1. H. L. Abbott and D. Hanson, *Lower bounds of certain types of van der Waerden numbers*, J. Combinatorial Theory **12** (1972), 143–146.
2. H. L. Abbott and L. Moser, *Sum-free sets of integers*, Acta Arith. **11** (1966), 393–396.
3. H. L. Abbott and D. Hanson, *On a problem of Schur and its generalizations*, Acta Arith. **20** (1972), 175–187.
4. L. D. Baumert, *Sum-free sets*, J. P. L. Research Summary No. 36–10, **1** (1961), 16–18.
5. S. L. G. Choi, *The largest sum-free subsequence from a sequence of n numbers*, Proc. Amer. Math. Soc. **39** (1973), 42–44.
6. ———, *On sequences not containing a large sum-free subsequence*, Proc. Amer. Math. Soc. **41** (1973), 415–418.
7. S. L. G. Choi, J. Komlós and E. Szemerédi, *On sum-free subsequences*, Trans. Amer. Math. Soc. **212** (1975), 307–313.
8. P. Erdős, *Extremal problems in number theory*, Proc. Sympos. Pure Math., vol. 8, Amer. Math. Soc., Providence, R. I., 1965, pp. 181–189.
9. H. Fredericksen, *Five sum-free sets*, Proc. Sixth Annual Southeastern Conf. on Graph Theory, Combinatorics and Computing, 1975, pp. 309–314.
10. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford Univ. Press, London and New York, 1960.
11. R. W. Irving, *An extension of Schur's theorem on sum-free partitions*, Acta Arith. **25** (1973), 55–63.
12. J. Komlós, M. Sulyok and E. Szemerédi, *Linear problems in combinatorial number theory*, Acta Math. Acad. Sci. Hungar. **26** (1975), 113–121.
13. L. Mirsky, *The combinatorics of arbitrary partitions*, Bull. Inst. Math. Appl. **11** (1975), 6–9.
14. I. Schur, *Über die Kongruenz $x^m + y^m \equiv z^m \pmod{p}$* , Jber. Deutsch. Math.-Verein. **25** (1916), 114–117.
15. W. D. Wallis, A. P. Street and J. S. Wallis, *Combinatorics: Room squares, sum-free sets, Hadamard matrices*, Lecture Notes in Math., vol. 292, Springer-Verlag, Berlin and New York, 1972.
16. E. G. Whitehead, Jr., *The Ramsey number $N(3, 3, 3, 3; 2)$* , Discrete Math. **4** (1973), 389–396.

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