GENERATORS FOR G BORDISM

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ABSTRACT. We show that a certain collection of G manifolds generates $N_*(G)$ as an algebra over N_* , the unoriented bordism ring, where G is finite abelian and of order not divisible by four.

1. Introduction. The object of this paper is to show that a certain natural collection of G manifolds multiplicatively generates the unoriented G bordism ring $N_*(G)$ as an algebra over the unoriented bordism ring N_* , for those finite abelian groups G such that $[G : 1]$ is not a multiple of four. In §§2 and 3 we will give a precise statement of the theorem for odd order groups and even order groups, respectively. For the remainder of the paper, all groups will be finite abelian and of order not divisible by four.

In the rest of this section, we will define terms, establish notation and recall results which we will need.

$\mathcal{F}$ (or $\mathcal{F}'$, $\mathcal{F}_1$, etc.) will denote some collection of subgroups of G , possibly empty. To be a "family in G ", $\mathcal{F}$ must have the property that $H \in \mathcal{F}$ and $K \subseteq H$, for K a subgroup of G , must imply that $K \in \mathcal{F}$. (For this definition and as a general reference, see [4].) $N_*(G; \mathcal{F}, \mathcal{F}')$ will denote the bordism algebra of smooth G actions on unoriented smooth manifolds, such actions having the property that the isotropy subgroups of points of M (resp. ∂M) lie in $\mathcal{F}$ (resp. $\mathcal{F}'$). If "All" denotes the family of all subgroups of G , then we denote $N_*(G; \text{All}, \phi)$ by $N_*(G)$. $N_*(G; \mathcal{F}, \mathcal{F}')(X)$, where X is a G space, is the bordism module of G maps of G manifolds (with the $(\mathcal{F}, \mathcal{F}')$ isotropy condition) into X .

We recall certain other notions of [4] which we need here. Two families $\mathcal{F} \supseteq \mathcal{F}'$ in G are called "adjacent" if they differ by a single group. There is the G -space $BO_k(G)$ and a k -plane bundle $E_k(G)$ with G action and projection $p_k: E_k(G) \rightarrow BO_k(G)$ which is a G map. These are formed as follows. Let $\{V_1, \dots, V_s\}$ be the collection of all irreducible real G representations. Let

$$V(G) = \left(\bigoplus_{i=1}^{\infty} (V_1)_i \right) \oplus \cdots \oplus \left(\bigoplus_{i=1}^{\infty} (V_s)_i \right),$$

where each $(V_j)_i$ is a copy of V_j . This has a natural orthogonal action of G . Hence the k planes through the origin in $V(G)$ also receive a G action.

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$BO_k(G)$ is this space. $E_k(G) \subseteq BO_k(G) \times V(G)$ and consists of those pairs (α, ν) with $\nu \in \alpha$. The diagonal action on $BO_k(G) \times V(G)$ restricts to $E_k(G)$ and the projection on the first factor gives p_k . This bundle is universal for k -plane bundles with G action over reasonable G spaces (certainly for manifolds). We will let $F'_H(BO_k(G))$ denote the points of $BO_k(G)$ which are left fixed by H (for H a subgroup of G) and over which $E_k(G)$ has no trivial H representation summand.

Then, according to [4, p. 20], for adjacent families $\mathcal{F} \supseteq \mathcal{F}'$, differing by the group H , there is an isomorphism

$$f_H: N_*(G; \mathcal{F}, \mathcal{F}') \rightarrow \bigoplus_{k=0}^* N_{*-k}(G/H; \{\{1\}\}, \phi)(F'_H(BO_k(G))).$$

The homomorphism assigns to a G action the classifying map of the normal bundle to the fixed set of the included H action. (Note that the above fixed point homomorphism sends the cartesian product algebra structure of the left-hand side to the multiplication induced by Whitney sums of vector bundles on the right-hand side.)

Finally, we say that a pair $(\mathcal{F}, \mathcal{F}')$ are “quasi-adjacent” with respect to a collection $\mathcal{Q} \subseteq \mathcal{F}$ if

- (i) $H \neq H'$ and H, H' in $\mathcal{Q}$ imply that $\text{grp}(H \cup H') \notin \mathcal{F}$;
- (ii) $L \in \mathcal{F} - \mathcal{F}'$ implies $\exists H \in \mathcal{Q}$ with $H \subseteq L$.

In this case, there is an isomorphism

$$N_*(G; \mathcal{F}, \mathcal{F}') \rightarrow \bigoplus_{H \in \mathcal{Q}} \bigoplus_{k=0}^* N_{*-k}(G/H, \mathcal{F}_H, \mathcal{F}'_H)(F_H BO_k(G)),$$

where $(\mathcal{F}_H, \mathcal{F}'_H)$ contain subgroups whose preimages are in $(\mathcal{F}, \mathcal{F}')$. See [4] as usual.

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2. Groups of odd order. Let $x(K)$ denote the bordism class of the factor group G/K with G action. $x(K) \in N_0(G)$ for every subgroup K in G . Let $\{\theta_i: i = 1, 2, \dots, r\}$ denote a complete set of nontrivial, irreducible complex representations of G , thought of as homomorphisms from G into $S^1 = U(1)$, the circle group. Let E_n denote the total space of the Hopf line bundle over $CP(n)$. Then G acts on E_n via θ_i . Let $Q(2n+2, i)$ denote the G manifold, $RP(E_n \oplus 1)$, with G action induced by $(\theta_i \oplus \text{trivial})$.

THEOREM 1. $N_*(G)$ is multiplicatively generated by $\{x(K): K \text{ is a subgroup of } G\}$ and $\{Q(2n+2, i): n = 0, 1, \dots; i = 1, 2, \dots, r\}$, as an algebra over N_* .

PROOF. First we construct an ascending chain of pairwise adjacent families in G from the empty family to All.

Let $\mathcal{P}_0 = \text{All} - \{G\}$. Having $\mathcal{P}_i$, let $\mathcal{P}_{i+1} = \mathcal{P}_i - \{\text{maximal subgroups in } \mathcal{P}_i\}$. Suppose $\mathcal{P}_n = \emptyset$. Now, for each i adjoin one at a time to $\mathcal{P}_{i-1}$, the

elements of $\mathcal{P}_i - \mathcal{P}_{i-1}$ (in any order) to get

$$\mathcal{P}_{i-1} = \mathcal{Q}_{i,1} \subsetneq \cdots \subsetneq \mathcal{Q}_{i,n_i} = \mathcal{P}_i.$$

The collection $\{\mathcal{Q}_{i,j}\}$ works.

If $\mathcal{F} \supseteq \mathcal{F}'$ are any successive families in this collection, which differ by K , then

$$N_*(G; \mathcal{F}, \mathcal{F}') = \bigoplus_{k=0}^* N_{*-k}(G/K; \text{Free})(F'_K(BO_k(G))),$$

where “Free” = $\{\{1\}\}$. This equals [4, p. 20]

$$\bigoplus_{k=0}^* N_{*-k}\left(E(G/K) \times_{(G/K)} F'_K(BO_k(G))\right),$$

where $E(G/K)$ denotes the total space of the universal principal G/K bundle. But, since G/K is an odd order group (also see §3),

$$E(G/K) \times_{(G/K)} F'_K(BO_k(G)) \approx B(G/K) \times F'_K(BO_k(G));$$

see [2, Proposition 3.2]. Hence, the above sum is

$$\bigoplus_{k=0}^* N_{*-k}(B(G/K) \times F'_K(BO_k(G))),$$

which is

$$N_*(B(G/K)) \otimes_{N_*} \left(\bigoplus_{k=0}^* N_{*-k}(F'_K(BO_k(G))) \right),$$

or just

$$\bigoplus_{k=0}^* N_{*-k}(f'_K(BO_k(G))),$$

since $N_*(B(G/K)) \cong N_*$, the unoriented bordism ring (G/K having odd order).

Since K has odd order, the nontrivial irreducible representations are all complex and $E_k(G)$ splits over $F'_K(BO_k(G))$ into a sum of complex bundles. Since $F'_K(BO_k(G))$ is a universal space, it must be a disjoint union of products $BU_{n_1} \times \cdots \times BU_{n_q}$, where q is the number of distinct nontrivial irreducible representations of K and $n_1 + \cdots + n_q = k/2$. K acts in the fibres over BU_{n_i} via its i th irreducible representation.

Putting all this together, then, there is an isomorphism of algebras

$$\rho: N_*(G; \mathcal{F}, \mathcal{F}') \rightarrow \bigoplus_{k=0}^{[* / 2]} \bigoplus_{(n)} N_{*-2k}(BU_{n_1} \times \cdots \times BU_{n_q}),$$

where (n) denotes the sequence $(n_1, \dots, n_r)$ and the sum is over all (n) with $n_1 + \cdots + n_q = k/2$.

We want to consider all exact sequences of the form

$$\cdots \rightarrow N_*(G; \mathcal{F}', \phi) \xrightarrow{i} N_*(G; \mathcal{F}, \phi) \xrightarrow{j} N_*(G; \mathcal{F}, \mathcal{F}') \rightarrow \cdots,$$

where i and j are induced by inclusions of families and $\partial(M, \psi) = (\partial M, \psi/\partial M)$. (See [4, p. 6], for a proof of exactness.) We assume that $\mathcal{F} \supseteq \mathcal{F}'$ are an adjacent pair of families from our previously constructed collection and that $N_*(G; \mathcal{F}', \phi)$ is generated over N_* by the products $[G/H, G][Q(m, i)]$, where $H \in \mathcal{F}'$. (Since $N_*(G; \text{Free}) \cong N_*$, our induction gets started.)

Let $K \in \mathcal{F} - \mathcal{F}'$. Note that

$$\rho(j[(G/K) \times Q(2n+2, i)]) = [CP(n) \subset BU_1].$$

As i varies over all the homomorphisms from G to S^1 , the actions of K on E_n will vary over all its irreducible representations. Hence, the inclusions $CP(n) \subset BU_1$ will vary over all the inclusions. But these maps generate (in fact, they are an algebra basis) the image of ρ .

It follows that $N_*(G; \mathcal{F}, \phi)$ is algebraically generated by the classes $[(G/K) \times Q(2n+2, i)]$, for $K \in \mathcal{F}$. This completes the proof.

3. $[G : 1] \equiv 2 \pmod{4}$. Write $G = H \times Z_2$, where H has odd order. Let $\{\theta_i; i = 1, 2, \dots, r\}$ denote a complete set of nontrivial irreducible complex representations of H as in §2. Let $P(n, k, i)$ denote $RP(E_n \oplus k \oplus 1)$, where the H action is induced by $(\theta_i \oplus \text{trivial} \oplus \text{trivial})$, and the Z_2 action is generated by $(-1) \oplus (-1) \oplus (1)$. Let $Q(2n+2, i)$ denote $RP(E_n \oplus 1)$ with H given by $\theta_i \oplus 1$ and trivial Z_2 action.

THEOREM 2. $N_*(G)$ is multiplicatively generated by $\{x(K); K \text{ is a subgroup of } G \text{ and } [G : K] \text{ is odd}\}$, $\{Q(2n+2, i); n = 0, 1, \dots; i = 1, 2, \dots, r\}$, $\{P(n, k, i); n = 0, 1, \dots; i = 1, 2, \dots, r; k = 0, 1, 2, \dots\}$ and the image of $N_*(Z_2)$ in $N_*(G)$ obtained by letting H act trivially on Z_2 manifolds.

PROOF. Let $\mathcal{F}$ denote $\{K \subseteq G; K \cap Z_2 = \{0\}\}$, which is a family of subgroups of G . There is an exact sequence

$$N_*(H \times Z_2) \xrightarrow{i_*} N_*(H \times Z_2; \text{All}, \mathcal{F}) \xrightarrow{\partial_*} N_*(H \times Z_2; \mathcal{F}).$$

Let (M, ϕ) denote an $\mathcal{F}$ -free $H \times Z_2$ action and give the interval $I = [-1, 1]$ an $H \times Z_2$ action by letting the generator of Z_2 act by multiplication by -1 and the elements of H act trivially. Then the twisted product $M \times_{(H \times Z_2)} I$ has a natural $(\text{All}, \mathcal{F})$ -free $H \times Z_2$ action and boundary (M, ϕ) . Since all is well defined on bordism, ∂_* is an epimorphism.

LEMMA 2.1. $N_*(H \times Z_2; \mathcal{F}) \cong N_*(H) \otimes_{N_*} N_*(Z_2; \text{Free})$ as N_* algebras.

PROOF. Let Δ denote the Smith homomorphism associated with the nontrivial 1-dimensional representation of G [3]; then there is an exact sequence:

$$\begin{array}{ccccc} N_*(H) & \xrightarrow{\text{extension}} & N_*(H \times Z_2; \mathcal{F}) & \xrightarrow{\Delta} & N_*(H \times Z_2; \mathcal{F}). \\ & \uparrow & \text{restriction} & \longleftarrow & \\ & & & & \end{array}$$

Since restriction is clearly zero, Δ is an epimorphism.

There is also the usual short exact sequence, where S denotes the usual Smith homomorphism:

$$0 \rightarrow N_* \rightarrow N_*(Z_2; \text{Free}) \xrightarrow{S} N_*(Z_2; \text{Free}) \rightarrow 0.$$

Tensoring this latter sequence with $N_*(H)$, a free N_* module and mapping it by the product homomorphism into the first sequence, we get:

$$\begin{array}{ccccccc} 0 \rightarrow N_*(H) & \rightarrow & N_*(Z_2; \text{Free}) \otimes N_*(H) & \xrightarrow{S \otimes 1} & N_*(Z_2; \text{Free}) \otimes N_*(H) & \rightarrow & 0 \\ \downarrow \cong & & \downarrow \otimes & & \downarrow \otimes & & \\ 0 \rightarrow N_*(H) & \rightarrow & N_*(H \times Z_2; \mathcal{F}) & \xrightarrow{\Delta} & N_*(H \times Z_2; \mathcal{F}) & \rightarrow & 0. \end{array}$$

An easy induction, using the Five-Lemma, completes the proof.

There are homomorphisms

$$N_*(H)(BO_1, \text{trivial}) \xrightleftharpoons[f]{g} N_*(H \times Z_2; \mathcal{F}),$$

which, it is easy to show using Lemma 2.1, are inverse isomorphisms. g is given by a sphere bundle construction and f classifies the free involution.

Since All and $\mathcal{F}$ are quasi-adjacent families in $H \times Z_2$ with respect to $\{Z_2\}$,

$$N_*(H \times Z_2; \text{All}, \mathcal{F}) = \bigoplus_{n=0}^* N_{*-n}(H)(BO_n(H))$$

as algebras and ∂_* becomes

$$\partial'_*: \bigoplus_{n=0}^* N_{*-n}(H)(BO_n(H)) \rightarrow N_*(H)(BO_1, \text{trivial})$$

given by the usual projective bundle construction.

Next we look at the (All, Free) exact sequence for the H bordism of the spaces $BO_n(H)$. We get

$$\begin{aligned} \cdots \rightarrow \bigoplus N_{*-n}(H; \text{Free})(BO_n(H)) &\xrightarrow{\alpha} \bigoplus N_{*-n}(H)(BO_n(H)) \\ &\xrightarrow{\beta} \bigoplus N_{*-n}(H; \text{All}; \text{Free})(BO_n(H)) \rightarrow \cdots, \end{aligned}$$

an exact sequence of algebras.

LEMMA 2.2. α is a monomorphism and the image of α is precisely the image of extension: $N_*(Z_2; \text{All}, \text{Free}) \rightarrow N_*(H \times Z_2; \text{All}, \mathcal{F})$.

PROOF. $N_*(H; \text{Free})(BO_n(H)) \cong N_*(EH \times_H BO_n(H))$. Clearly $EH \times_H E_n(H)$ is an n -plane bundle over $EH \times_H BO_n(H)$. Classifying this and projecting to BH gives a homotopy equivalence

$$EH \times_H BO_n(H) \rightarrow BH \times BO_n.$$

Since H has odd order, $N_*(BH \times BO_n) \cong N_*(BO_n)$.

The composite isomorphism

$$j_*: \bigoplus N_{*-n}(H; \text{Free})(BO_n(H)) \rightarrow \bigoplus N_{*-n}(BO_n)$$

takes the n -plane bundle η over M to η/H over M/H .

There is also the extension homomorphism e which sends an n -plane bundle η over M to $\eta \times H$ over $M \times H$. Since j_*e is the identity, e is an isomorphism.

There is also a forgetful homomorphism

$$\rho: \bigoplus N_{*-n}(H)(BO_n(H)) \rightarrow \bigoplus N_{*-n}(BO_n)$$

with $\rho\alpha = j$. Hence, α is a monomorphism and image α is Z_2 extensions. This proves 2.2.

Therefore, $N_*(H \times Z_2; \text{All}, \mathcal{F})$, modulo the ideal of Z_2 extensions (from $N_*(Z_2; \text{All}, \text{Free})$), is isomorphic as N_* algebras to $\bigoplus_{n=0}^* N_{*-n}(H; \text{All}, \text{Free})(BO_n(H))$. In $N_*(H \times Z_2)$, the ideal of Z_2 extensions is given by products of Z_2 classes with $x(Z_2)$. Let

$$\mathcal{F}_r = \text{All} \supsetneq \cdots \supsetneq \mathcal{F}_{i+1} \supsetneq \mathcal{F}_i \supsetneq \cdots \supsetneq \mathcal{F}_1 = \text{Free}$$

be a sequence of successively adjacent families in H . We wish to consider all sequences

$$\begin{aligned} \cdots \rightarrow N_*(H; \mathcal{F}_i, \mathcal{F}_1)(BO_n(H)) &\xrightarrow{\gamma} N_*(H; \mathcal{F}_{i+1}, \mathcal{F}_1)(BO_n(H)) \\ &\xrightarrow{\delta} N_*(H; \mathcal{F}_{i+1}, \mathcal{F}_i)(BO_n(H)) \rightarrow \cdots \end{aligned}$$

and wish to establish the claim that $N_*(H; \mathcal{F}_{i+1}, \mathcal{F}_1)(BO_n(H))$ is generated by our potential generators (together with classes which split back from $N_*(H \times Z_2, \mathcal{F})$) if $N_*(H; \mathcal{F}_i, \mathcal{F}_1)$ is. (Since $\mathcal{F}_2$ differs from $\mathcal{F}_1$ by a single group, the proof of the induction step will also get the induction started.)

LEMMA 2.3.

$$\begin{aligned} \bigoplus_{n=0}^* N_{*-n}(H; \mathcal{F}_{i+1}, \mathcal{F}_i)(BO_n(H)) &\cong \left(\bigoplus_{r=0}^* N_{*-r}(F'_K BO_r(H)) \right) \\ &\otimes \left(\bigoplus_{s=0}^* N_{*-s}(F'_K BO_s(H)) \right) \otimes \left(\bigoplus_{t=0}^* N_{*-t}(BO_t) \right), \end{aligned}$$

where $\{K\} = \mathcal{F}_{i+1} - \mathcal{F}_i$.

PROOF. According to [4], $N_{*-n}(H; \mathcal{F}_{i+1}, \mathcal{F}_i)(BO_n(H))$ is isomorphic to

$$\bigoplus_{k=0}^{*-n} N_{*-n-k}(H/K; \text{Free})(F_K BO_n(H) \times F'_K BO_k(H)),$$

which is

$$\bigoplus_{k=0}^{*-n} N_{*-n-k}((F_K BO_n(H) \times F'_K BO_k(H)) \times_{H/K} E(H/K)).$$

It is possible to change the H action in the universal bundle over $F_K BO_n(H) \times F'_K BO_k(H)$ to make K act trivially and for the resulting H/K action to cover the base. Hence,

$$\begin{aligned} \bigoplus_{n=0}^* N_{*-n}(H; \mathcal{F}_{i+1}, \mathcal{F}_i)(BO_n(H)) \\ \cong \bigoplus_{n=0}^* \bigoplus_{k=0}^{*-n} N_{*-n-k}(F_K BO_n(H) \times F'_K BO_k(H)). \end{aligned}$$

It follows from §1 that $F_K BO_n(H)$ is a disjoint union of all products $BO_t \times BU_{n_1} \times \cdots \times BU_{n_p}$, where $t + 2n_1 + \cdots + 2n_p = n$ and p is the number of distinct nontrivial irreducible complex representations of K . Let $(n) = (n_1, n_2, \dots, n_p)$ and $|n| = n_1 + \cdots + n_p$; then

$$F_K BO_n(H) = \bigcup_{s+2|n|=n} BO_s \times BU_{(n)}$$

and

$$F'_K BO_k(H) = \bigcup_{2|q|=k} BU_{(q)}.$$

The Künneth Theorem finishes the proof of 2.3.

Denote $Q(2n+2, i)$ by $x(n, i)$ and $P(n, k, i)$ by $y(n, k, i)$. There are the following classes in

$$\begin{aligned} A_*(K) = \left(\bigoplus_{j=0}^* N_{*-j}(F'_K BO_j(H)) \right) \otimes \left(\bigoplus_{k=0}^* N_{*-k}(F'_K BO_k(H)) \right) \\ \otimes \left(\bigoplus_{l=0}^* kN_{*-l}(BO_l) \right). \end{aligned}$$

The first factor is generated by the bundle-bordism classes $\alpha(n, j) =$ class of the Hopf bundle over $CP(n)$ with H acting via θ_j . The second factor is generated by $\beta(n, j)$ defined exactly as $\alpha(n, j)$, and the third factor is generated by $\gamma(n) =$ class of the twisted line bundle over $RP(n-1)$. Finally, we need the following convenient generating set for $N_*(Z_2)$ [1]. Let $z(j)$ be the class of the manifold $RP(j)$ with involution, $[r_0; r_1; \cdots; r_j] \rightarrow [-r_0; r_1; \cdots; r_j]$ and K_1 the N_* endomorphism on $N_*(Z_2)$ given on Z_2 manifolds by $K_1(M, T) = ((M \times S^1)/(T \times (-1)), 1 \times \text{conjugation})$. It is known that the classes $(K_1^k z(j_1))z(j_2) \cdots z(j_n) = z(j_1, \dots, j_n, k)$ generate $N_*(Z_2)$.

It is clear that $A_*(K)$ is generated as a N_* module by monomials of the form $\gamma(1)^k \gamma(h_1) \cdots \gamma(h_n) \alpha(n_1, j_1) \cdots \alpha(n_m, j_m) \beta(m_1, l_1) \cdots \beta(m_p, l_p)$, where $k \geq 0$; $1 < h_1 \leq h_2 \leq \cdots \leq h_n$. (We include the possibility that n (resp. m or p) may be zero, in which case no $\gamma(h_j)$ (resp. α or β) may appear.)

Consider the following collection of cases:

1. $n \neq 0$;
2. $n = 0$ and $k = 0$;
3. $n = 0, k \neq 0, p = 0$;
4. $n = 0, k \neq 0, p \neq 0$.

Case 1. Consider the class

$$\beta i_* \left[x(K) z(h_1, \dots, h_n, k) \left(\prod_{i=1}^m x(n_i, j_i) \right) \left(\prod_{i=1}^p y(m_i, 0, l_i) \right) \right].$$

Its image in $A_*(K)$ is [1]

$$(\gamma(1)^k \gamma(h_1) \cdots \gamma(h_n) + \Gamma)(\alpha(n_1, j_1) \cdots \alpha(n_m, j_m)) \\ \times (\beta(m_1, l_1) \cdots \beta(m_p, l_p)),$$

where Γ is a sum of monomials of the form $\gamma(1)^r \gamma(s_1) \cdots \gamma(s_r)$ where $r > k$. These images therefore generate this case.

Case 2.

$$\beta i_* \left[x(K) \left(\prod_{i=1}^m x(n_i, j_i) \right) \left(\prod_{i=1}^p y(m_i, 0, l_i) \right) \right]$$

generate this case.

Case 3. By Lemma 2.1, these classes split back from $N_*(H \times Z_2; \mathfrak{F})$.

Case 4. We want to show that we can hit classes of the form:

$$\gamma(1)^k \alpha(n_1, j_1) \cdots \alpha(n_m, j_m) \beta(m_1, l_1) \cdots \beta(m_p, l_p)$$

where $p \geq 1$. We induct on p . If $p = 1$, we consider the class

$$\beta i_* \left[x(K) \left(\prod_{i=1}^m x(n_i, j_i) \right) y(m_1, k, l_1) \right]$$

in $A_*(K)$, obtaining $(\gamma(1)^k \beta(m_1, l_1) + \Gamma) \alpha(n_1, j_1) \cdots \alpha(n_m, j_m)$, where Γ is a line bundle over $\mathbf{R}P(E_{m_1} \oplus k)$. Hence we can hit this class with the help of classes from $N_*(H \times Z_2; \mathfrak{F})$. For $p > 1$,

$$i_* \left[x(K) \left(\prod_{i=1}^m x(n_i, j_i) \right) y(m_1, k, l_1) \left(\prod_{i=2}^p y(m_i, 0, l_i) \right) \right]$$

is, in $A_*(K)$,

$$\prod_{i=1}^m \alpha(n_i, j_i) \gamma(1)^k \beta(m_1, l_1) \prod_{i=2}^p \beta(m_i, l_i) + \Gamma \prod_{i=1}^m \alpha(n_i, j_i) \prod_{i=2}^p \beta(m_i, l_i).$$

Since $\Gamma = \sum_{i+j=1+k+2m} a_i \gamma(1)^j$, where $a_i \in (N_*)_i$, the induction hypothesis and Case 3 finishes this case.

We have shown that monomials in our proposed generators along with those which split back from $N_*(H \times Z_2; \mathfrak{F})$ will generate $N_*(H \times Z_2; \text{All}, \mathfrak{F})$ as an N_* module. But $N_*(G; \text{All}, \mathfrak{F}) \cong N_*(G) \oplus N_*(G; \mathfrak{F})$ via this splitting. This proves Theorem 2.

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