EXTREMAL PROPERTIES OF REAL AXIALLY SYMMETRIC HARMONIC FUNCTIONS IN E^3

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*This paper is dedicated to the memory of Professor Stefan Bergman
from whose work we have learned so much*

ABSTRACT. The set $\mathcal{H}$ consists of all real harmonic functions defined by $U(r, \theta) = \sum_{k=0}^{\infty} a_k r^k P_k(\cos \theta)$ which are regular in Σ , the open unit sphere about the origin E^3 . Two problems arise concerning $\mathcal{H}$ and a subset $\mathcal{H}_*$ whose members U have the first $n+1$ coefficients $a_0, \dots, a_n$ specified. (1) For $U \in \mathcal{H}$, determine $I(U) = \inf\{U(r, \theta) | (r, \theta) \in \Sigma\}$ as the limit of a monotone sequence of constants $\{\lambda_k(a_0, \dots, a_k)\}_{k=0}^{\infty}$ which can be computed algebraically. (2) Find $U_0 \in \mathcal{H}_*$ and the constant

$$\lambda_n(a_0, \dots, a_n) = \sup\{I(U) | U \in \mathcal{H}_*\} = I(U_0).$$

These are answered by means of the Bergman Integral Operator Method and applications of the Methods of Ascent and Descent to the classical Carathéodory-Fejér problem regarding extremal properties of harmonic functions in E^2 .

1. Introduction. In the theory of analytic functions

$$(1.1) \quad f(z) = a_0 + 2 \sum_{k=1}^{\infty} a_k z^k$$

of one complex variable, the unique relation between f and its Taylor's coefficients has been exploited to provide definitive characterizations of three fundamental properties of f ;

(i) the location and nature of the singularities [Hadamard, Mandelbrojt (see [3])]

(ii) the distribution of values [Cauchy, Carathéodory-Toeplitz and Schur, (see [7], [11])]

(iii) and extremal properties of certain functionals defined for f analytic in the open unit disk D about the origin [Carathéodory-Fejér, (see [6])]. These are translated to statements on real harmonic functions (HF)

$$(1.2) \quad u(x, \rho) = a_0 + 2 \sum_{k=1}^{\infty} a_k u_k(x, \rho),$$

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$$(1.3) \quad (x + i\rho)^k = u_k(x, \rho) + iv_k(x, \rho)$$

in E^2 by taking the real part of the associated analytic function $f(x + i\rho)$. In E^3 , one anticipates that similar characterizations of axially symmetric harmonic functions (AHF)

$$(1.4) \quad U(x, \rho) = a_0 + 2 \sum_{k=1}^{\infty} a_k U_k(x, \rho),$$

$$(1.5) \quad U_k(x, \rho) = (x^2 + \rho^2)^{k/2} P_k\left(x/\sqrt{x^2 + \rho^2}\right),$$

$(x, y, z) \in E^3$, $y^2 + z^2 = \rho^2$, $x^2 + \rho^2 = r^2$, $\cos \theta = xr^{-1}$ require operators whose role is analogous to that of the operator Re which represents a HF in E^2 for which a specific property is sought as the transform of an analytic function whose corresponding property is known. Application of function theoretic methods to integral representations of AHF based on the LaPlace type integral for the zonal harmonics (1.5) verified this for the first two properties.

A definitive characterization of the singularities of AHF is provided by Gilbert's theorems [4], [5] where integral operators and the "Envelope Method" extended the classical Hadamard and Mandelbrojt theorems. Moreover, an extensive literature on the singularity problem for other types of orthogonal expansions may be found in [2] and [4], [5].

The distribution of values of AHF are generalizations of the theorems in (ii). This was shown by M. Marden [8] and the author [9], [10] who applied convexity arguments from the analytic theory of polynomials of one complex variable to Bergman-Whittaker [1] and Gilbert type [4] integral representations of AHF.

This paper establishes that known properties of the functional $i(u) = \lim_{r \uparrow 1} i_r(u)$,

$$(1.6) \quad i_r(u) = \inf\{u(x, \rho) | x^2 + \rho^2 < r^2\}$$

defined for $u \in \mathcal{H}$, the class of real HF in D , also characterize the functional $I(U) = \lim_{r \uparrow 1} I_r(U)$,

$$(1.7) \quad I_r(U) = \inf\{U(x, \rho) | x^2 + \rho^2 < r^2\}$$

defined for $U \in \mathcal{H}$, the class of real AHF in the open unit sphere about the origin in E^3 . Basic to this are the integral operators which we now consider.

2. Basic formulas. The operators employed here, with the Bergman, Gilbert and Whittaker operators, are based on the LaPlace integral for the zonal harmonics which we write as

$$(2.1) \quad U_k(x, \rho) = \frac{1}{2\pi} \int_0^{2\pi} [u_k(x, \rho \cos t) + iv_k(x, \rho \cos t)] dt.$$

Visibly v_k , the conjugate harmonic function to u_k , is in the null space of the operator. This suggests the operator $U(x, \rho) = W[u(x, \rho)]$ mapping (1.2) onto (1.4) by

$$(2.2) \quad U(x, \rho) = \frac{1}{2\pi} \int_0^{2\pi} u(x, \rho \cos t) dt.$$

Evidently, if u is harmonic for $x^2 + \rho^2 < R^2$, the same is true of U .

The inverse operator uses orthogonality of the Legendre polynomials to relate the harmonic polynomials by the transform

$$(2.3) \quad u_k(x, \rho) = \int_{-1}^1 U_k(r\eta, r(1 - \eta^2)^{1/2}) d\eta,$$

$$(2.4) \quad d\nu = K(xr^{-1}, \rho r^{-1}, \eta) d\eta$$

whose kernel $K(xr^{-1}, \rho r^{-1}, \eta) = \operatorname{Re} K(z, \eta)/2$ corresponds to the generating function

$$(2.5) \quad K(z, \eta) = \sum_{k=0}^{\infty} (2k+1)z^k P_k(\eta) = (1 - z^2)/(1 - 2\eta z + z^2)^{3/2}$$

[1, p. 5] for the ultraspherical polynomials $C_k^{1/2}(\eta) = P_k(\eta)$ [1, p. 24] with complex $z = (x + i\rho)r^{-1} = \operatorname{Re}[R \exp(i\phi)]$. The kernel is nonsingular if $R < 1$. Consequently, the inversion formula $u(x, \rho) = W^{-1}[U(x, \rho)]$ associating (1.2) and (1.4) by

$$(2.6) \quad u(x, \rho) = \int_{-1}^1 U(r\eta, r(1 - \eta^2)^{1/2}) d\nu(xr^{-1}, \rho r^{-1}, \eta)$$

represents a HF in $x^2 + \rho^2 < R^2$ where U is harmonic in $r < R$. By construction, each real AHF is represented uniquely as the transform of a real HF and conversely. By applying definitions [1, p. 21; p. 7] $C_k^{1/2}(\eta) = P_k^{(0,0)}(\eta)$, $P_k^{(0,0)}(1) = 1$ and $P_k^{(-1/2, -1/2)}(\cos \theta)/P_k^{(-1/2, -1/2)}(1) = \cos k\theta$ to Theorem 3.3 [1, p. 25] one writes the kernel as

$$(2.7) \quad K(z, \eta) = \int_{-1}^1 T(z, \eta') d\mu_\eta(\eta'), \quad d\mu_\eta(\eta') \geq 0,$$

$$(2.8) \quad \operatorname{Re}[T(z, \cos \theta)] = \sum_{k=0}^{\infty} (k + 1/2) R^k \cos k\phi \cos k\theta.$$

However, (2.8) being the generalized translation [1, p. 34, Theorem 4] of

$$(2.9) \quad \operatorname{Re}[T(z, 1)] = \frac{(1 - R^2)(1 - R \cos \phi) - 2R^2(\sin^2 \phi)}{|1 - R^2 + 2R \cos \phi|^2},$$

a nonnegative function for $0 \leq \phi \leq 2\pi$, $0 \leq R \leq R_0$, R_0 being the smallest positive zero of $R^3 - 3R^2 + 1 = 0$, is necessarily nonnegative for $0 \leq \theta$, $\phi < 2\pi$, $0 \leq R \leq R_0$ as is the kernel by (2.7). We now turn to

3. The extremal properties. Our first result, an analogy with Carathéodory-Fejér [6, pp. 147, 151], concerns the infimum of AHF in Σ .

THEOREM 1. Let $U \in \mathcal{H}$ be expanded as

$$(3.1) \quad U(x, \rho) = a_0 + 2 \sum_{k=1}^{\infty} a_k U_k(x, \rho)$$

and $\lambda_k = \lambda_k(a_0, \dots, a_k)$ be the smallest eigenvalue of the Toeplitz matrix T_k

$$(3.2) \quad T_k(a_0, \dots, a_k) = \begin{bmatrix} a_0 & a_1 & \cdots & a_k \\ a_1 & a_0 & \cdots & a_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_k & a_{k-1} & \cdots & a_0 \end{bmatrix}$$

then the sequence $\{\lambda_k\}_{k=0}^\infty$ is monotone decreasing and

$$(3.3) \quad I(U) = \lim_{k \rightarrow \infty} \lambda_k.$$

PROOF. Let $U \in \mathcal{H}$ and $u \in \mathcal{h}$ such that $U(x, \rho) = W[u(x, \rho)]$. Then since $W[1] = 1$, $U(x, \rho) \geq i(u)$, $x^2 + \rho^2 < 1$ and

$$(3.4) \quad I(U) \geq i(u).$$

To establish the reverse inequality, note homogeneity of the harmonic polynomials

$$(3.5) \quad \begin{aligned} U_k(x, \rho) &= R_0^k U_k(x R_0^{-1}, \rho R_0^{-1}), \\ u_k(x, \rho) &= R_0^k u_k(x R_0^{-1}, \rho R_0^{-1}) \end{aligned}$$

in defining the HF by means of a homothetic transformation on (1.2) (1.4) as

$$(3.6) \quad \begin{aligned} U_*(x, \rho) &= a_0 + 2 \sum_{k=1}^{\infty} a_k R_0^k U_k(x, \rho), \\ u_*(x, \rho) &= a_0 + 2 \sum_{k=1}^{\infty} a_k R_0^k u_k(x, \rho) \end{aligned}$$

in $x^2 + \rho^2 < R_0^2$ so that $I(U) = I_{R_0}(U_*)$ and $i(u) = i_{R_0}(u_*)$. The inverse operator is normalized by $W^{-1}(1) = 1$ and the kernel is nonnegative in $r < R_0$. Thus, $u_*(x, \rho) = W^{-1}[U_*(x, \rho)] \geq I_{R_0}(U_*)$ and $i_{R_0}(u_*) \geq I_{R_0}(U_*)$. In view of the relation between U_* and U , and u_* and u , we find $i(u) \geq I(U)$ to conclude

$$(3.7) \quad i(u) = I(U).$$

Citing [6, p. 147] completes the reasoning. Next consider an extremal property of $I(\cdot)$ on a subset $\mathcal{H}(a_0, \dots, a_n) \subset \mathcal{H}$ of all those AHF in $\mathcal{H}$ whose first $n+1$ coefficients $a_0, \dots, a_n$ are fixed.

THEOREM 2. Let $U \in \mathcal{H}(a_0, \dots, a_n)$ be expanded as in (3.1) and $\lambda_n = \lambda_n(a_0, \dots, a_n)$ be the smallest eigenvalue of the Toeplitz matrix $T_n(a_0, \dots, a_n)$. Then

$$(3.8) \quad I(U) \leq \lambda_n$$

and

$$(3.9) \quad \sup\{I(u) | U \in \mathcal{H}(a_0, \dots, a_n)\} = I(U_0)$$

for unique $U_0 \in \mathcal{H}(a_0, \dots, a_n)$ expanded as

$$(3.10) \quad U_0(x, \rho) = 2\lambda_n + 2 \sum_{k=1}^m \mu_k V_k(x, \rho) - U(0, 0),$$

$$(3.11) \quad V_k(x, \rho) = \operatorname{Re} \left[\varepsilon_k / \sqrt{(x - \varepsilon_k)^2 + \rho^2} \right].$$

The principal branch is taken and ε_k , $|\varepsilon_k| = 1$, $\mu_k > 0$ and $1 \leq m \leq n$ are unique. If $a_0^2 + \dots + a_n^2 = 0$, the extremal function reduces to $U_0(x, \rho) = a_0 = \lambda_n$.

PROOF. For the set $\mathcal{H}_* = \mathcal{H}(a_0, \dots, a_n)$ one associates the set $\mathcal{K}_* = \mathcal{K}(a_0, \dots, a_n) = \{W^{-1}U | U \in \mathcal{H}_*\}$. Because of (3.7), the unique extremal function $u_0 \in \mathcal{H}_*$ for $i(\cdot)$ is mapped by W onto the unique extremal function $U_0 \in \mathcal{H}_*$ for $I(\cdot)$. The function

$$U_0(x, \rho) = W[u_0(x, \rho)], \quad u_0(x, \rho) = \operatorname{Re} f(x + i\rho)$$

where f is defined by [6, p. 151, #2].

4. Generalization. The corresponding problem of AHF on axisymmetric domains Ω [8], [9] unlock by mapping the associated axiconvex set ω into D in a way similar to that in [10]. The Carathéodory-Fejér properties referring to supremum of HF in E^2 extend to AHF by replacing inf by sup and interpreting the associates as in [6] while reversing the inequalities. Extension to AHF in $E^{2\alpha+1}$, $2\alpha + 1 > 0$, follows from operators using the normalized ultraspherical polynomials [1, p. 32].

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