SIMPLE CONTINUED FRACTIONS AND SPECIAL RELATIVITY

DOUGLAS HENSLEY¹

ABSTRACT. Let $E_0, E_1, \dots, E_n$ be inertial frames of reference in a one dimensional relativistic universe where the speed of light is $c = \sqrt{k}$, k some natural number. For $n > 1$ let E_n have velocity 1 with respect to E_{n-1} . Let x_n denote the velocity of E_n with respect to E_0 . Then only if $k = 2, 3$ or 5 will x_n be a simple continued fraction convergent of $\sqrt{k}$ infinitely often.

Introduction. Let c denote the speed of light, and let E_0, E_1 and E_2 be inertial frames of reference with collinear velocities. Let the velocity of E_1 with respect to E_0 be x , and of E_2 with respect to E_1 , y . The theory of special relativity then gives the velocity of E_2 with respect to E_0 as $(x + y)/(1 + xyc^{-2})$, which we denote $x \oplus_c y$.

Let units of measurement be chosen so that c^2 is an integer. Given x_0 , let x_n denote the sum $x_0 \oplus_c 1 \oplus_c 1 \cdots \oplus_c 1$ (n additions of 1), so that $x_n = x_{n-1} \oplus_c 1$ if $n > 1$. If x_0 is rational, the sequence (x_n) will be a sequence of rational numbers converging to the square root of the integer c^2 .

We ask whether x_n is infinitely often a simple continued fraction convergent of c . K. B. Stolarsky noted that if $x_0 = 1$ and $c = \sqrt{2}$ or $\sqrt{3}$, this is the case, and posed the more general problem. First let us note one more example. If $c = \sqrt{5}$ and $x_0 = 1$ then x_{6n+2} is the $2n$ th convergent of $\sqrt{5}$. In what follows we show why there are no other examples. I wish to thank Professor Freeman Dyson for his helpful comments.

THEOREM. *If $c^2 = k$ is a nonsquare integer > 5 and if $x_0 \in Q$ then there are at most finitely many n such that x_n is a continued fraction convergent of $\sqrt{k}$.*

PROOF. Let $a = a_0 + a_1\sqrt{k}$ be such that $a_0, a_1 \in \mathbf{Z}$ and $x_0 = \sqrt{k} (a - \bar{a})/(a + \bar{a})$, where conjugation is in the field $Q(\sqrt{k})$. Let $\alpha = k + \sqrt{k}$.

LEMMA. *If $x = \sqrt{k} (b - \bar{b})/(b + \bar{b})$, $b \in Q(\sqrt{k})$, then $x \oplus_c 1 = \sqrt{k} (b\alpha - \bar{b}\bar{\alpha})/(b\alpha + \bar{b}\bar{\alpha})$.*

PROOF. Let $b = b_0 + b_1\sqrt{k}$ with $b_0, b_1 \in Q$. Then

$$x \oplus_c 1 = (b_1k/b_0 + 1)/(1 + b_1/b_0) = (b_1k + b_0)/(b_1 + b_0).$$

Received by the editors April 22, 1977.

AMS (MOS) subject classifications (1970). Primary 10A30, 10A35, 10F20, 83A05.

Key words and phrases. Special relativity, simple continued fraction, velocity addition laws.

¹Supported in part by NSF grant MCS72-05055 A04.

© American Mathematical Society 1978

Likewise

$$\begin{aligned}\sqrt{k} (b\alpha - \bar{b}\bar{\alpha}) / (b\alpha + \bar{b}\bar{\alpha}) &= \sqrt{k} ((b_1k + b_0)\sqrt{k} / (b_0k + b_1k)) \\ &= (b_1k + b_0) / (b_1 + b_0). \quad \square\end{aligned}$$

COROLLARY. If x_0 , a and α are as above, then

$$x_n = \sqrt{k} (a\alpha^n - \bar{a}\bar{\alpha}^n) / (a\alpha^n + \bar{a}\bar{\alpha}^n).$$

PROOF. Take $b = a\alpha^{n-1}$. Then $\bar{b} = \bar{a}\bar{\alpha}^{n-1}$ and the induction step follows from the Lemma above. And by assumption the claim holds when $n = 0$. $\square$

Recall that for each nonsquare k there exists an M such that if $(d_1, d_2) = 1$ and d_1/d_2 is a continued fraction convergent of $\sqrt{k}$, then $|d_1^2 - kd_2^2| < M$ [1].

Now suppose that in $Q(\sqrt{k})$ there is some prime ideal p which divides $\bar{\alpha}/\alpha$ to a positive power. This happens if and only if $\alpha/\bar{\alpha}$ is not an integer of $Q(\sqrt{k})$. If for some n , $x_n = (d_1/d_2)$ is a continued fraction convergent of $\sqrt{k}$ then by the Corollary,

$$\sqrt{k} (a\alpha^n - \bar{a}\bar{\alpha}^n) / (a\alpha^n + \bar{a}\bar{\alpha}^n) = d_1/d_2$$

so that

$$k(a\alpha^n - \bar{a}\bar{\alpha}^n)^2 d_2^2 - (a\alpha^n + \bar{a}\bar{\alpha}^n)^2 d_1^2 = 0,$$

and

$$k(a - \bar{a}(\bar{\alpha}/\alpha)^n)^2 d_2^2 - (a + \bar{a}(\bar{\alpha}/\alpha)^n)^2 d_1^2 = 0.$$

Since p^n divides $(\bar{\alpha}/\alpha)^n$, p^n divides $a^2(kd_2^2 - d_1^2)$. Since $kd_2^2 - d_1^2$ is a nonzero integer with absolute value $< M$ it contains only finitely many powers of p . Therefore there can be only finitely many n for which x_n is a continued fraction convergent of $\sqrt{k}$.

To complete the proof we note that $\alpha/\bar{\alpha}$ is an integer of $Q(\sqrt{k})$ if and only if the polynomial

$$(x - \alpha/\bar{\alpha})(x - \bar{\alpha}/\alpha) = x^2 - 2(k+1)x/(k-1) + 1$$

has integer coefficients, and that this occurs exactly when $k = 2, 3$ or 5 . $\square$

BIBLIOGRAPHY

1. I. Niven and H. S. Zuckerman, *An introduction to the theory of numbers*, 2nd ed., Wiley, New York, 1966, Theorems 7.19 and 7.22.

SCHOOL OF MATHEMATICS, THE INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540

Current address: Department of Mathematics, Texas A & M University, College Station, Texas 77843