NEW PROOFS OF CHAPMAN'S CE MAPPING THEOREM AND WEST'S MAPPING CYLINDER THEOREM

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ABSTRACT. This note proves two theorems of the theory of Q -manifolds; the original proofs are simplified by using Bing shrinking criterion.

1. We give new proofs of two results¹ in the theory of Hilbert cube manifolds (" Q -manifolds"): T. A. Chapman's CE mapping theorem and J. West's first mapping cylinder theorem. They respectively assert that a CE (= cell-like) map between Q -manifolds is a near-homeomorphism, and that the mapping cylinder of a map between Q -manifolds is a Q -manifold factor (see below for precise statements). M. Handel has also a new proof of West's mapping cylinder theorem [6].

Following R. D. Edwards, we consider these problems in light of the notion of *shrinkability* (see [10] for an exposition of the Bing shrinking criterion). We naturally use the most elementary facts about Q -manifolds, essentially the five first chapters of Chapman's set of notes [2], and make current use of the (now) classical notions of ANR's and cell-like mappings—see, respectively, [8] and [13] and below.

We put on spaces of continuous maps the fine topology. More precisely a basis for a neighborhood of $f \in C(X, Y)$ is given by the set $N(f, \mathcal{U})$, where $\mathcal{U}$ is an open covering of Y and $N(f, \mathcal{U}) = \{g \mid \forall x \in X \exists U \in \mathcal{U} \text{ such that } f(x) \text{ and } g(x) \in U\}$. If g is in $N(f, \mathcal{U})$, we say that f and g are $\mathcal{U}$ -close. A map is a near-homeomorphism if it is arbitrarily closely approximable by homeomorphisms.

We need first some definitions.

A *Hilbert cube manifold* (or Q -manifold) is a separable metric space locally homeomorphic to the countable product of closed intervals, the Hilbert cube $Q = \prod_{n=1}^{\infty} I_n$.

A map $f: X \rightarrow Y$ between two locally compact spaces X and Y is *proper* if it is closed and all point-inverses $f^{-1}(y)$, y in Y , are compact. The *mapping*

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¹(Added January 1976.) They are now trivial corollaries of the following recent theorem of R. D. Edwards: Let $f: M \rightarrow X$ be a CE map from a Q -manifold M onto an ANR X . Then $f \times \text{id}: M \times Q \rightarrow X \times Q$ is a near-homeomorphism [14].

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cylinder $Z(f)$ of f is the space obtained from the disjoint union $X \times [0, 1] \sqcup Y$ by identifying $(x, 1)$ in $X \times \{1\}$ to $f(x)$ in Y .

By *ANR* (= Absolute Neighborhood Retract), we mean here a locally compact separable metric space which has the following property: if embedded as a *closed* set in a metric space, it is the retract of one of its neighborhoods. The property of being an ANR is stable by taking products and mapping cylinders [8]; it is *hereditary* for open sets and *local*, in the sense that a space is an ANR if and only if each point has a neighborhood which is an ANR. The Hilbert cube, and so any Q -manifold, are ANR's.

A map $f: X \rightarrow Y$ between ANR's is *cell-like* (= CE) provided it is proper, onto, and each point-inverse $f^{-1}(y)$ has property UV^∞ in X : this means that for each neighborhood U of $f^{-1}(y)$, there exists a neighborhood $V \subset U$ such that the inclusion $V \hookrightarrow U$ is homotopic to a constant map of V in U . For example, if all the $f^{-1}(y)$ are contractible, f must be CE. We will use the following fundamental result (see [7] for a proof).

THEOREM. *A map $f: X \rightarrow Y$ between ANR's is CE if and only if it is a fine homotopy equivalence, i.e. f is proper, onto, and for each open cover $\mathcal{U}$ of Y there exists a proper map $g: Y \rightarrow X$ such that gf is proper homotopic to the identity of X by a (proper) homotopy limited by the open cover $f^{-1}(\mathcal{U}) = \{f^{-1}(U), U \in \mathcal{U}\}$, and fg is homotopic to the identity of Y by a proper homotopy limited by $\mathcal{U}$.*

As a corollary, any finite composition of CE maps between ANR's is a CE map.

Bing's shrinkability criterion offers a necessary and sufficient condition that a proper surjective mapping $p: X \rightarrow Y$ of a complete metric space X onto a space Y be a near-homeomorphism. The criterion requires roughly that the compacta of the induced *decomposition* $\{p^{-1}(y), y \in Y\}$ of X should be simultaneously shrinkable to arbitrary small size by self-homeomorphisms of X . We need only the locally compact case, which is proved in [14, VII].

THEOREM. *A proper surjection $p: X \rightarrow Y$ of locally compact spaces is a near-homeomorphism if and only if for each open cover $\mathcal{U}$ of X and $\mathcal{V}$ of Y , there exists a "shrinking" homeomorphism $h: X \rightarrow X$ such that ph is $\mathcal{V}$ -close to p and for each y in Y , $hp^{-1}(y)$ lies in some element of $\mathcal{U}$.*

A closed set A in a space X is said to be a Z -set in X provided that the identity of X can be arbitrarily closely approximated by maps of X into $X - A$. Z -sets in ANR's can also be defined by the "locally homotopy negligible" property, as explained in [11]. For Z -sets in Q -manifolds, we have the following fundamental *unknotting* result [14, IV]:

THEOREM. *Let M be a Q -manifold, A be locally compact and let $F: A \times I \rightarrow M$ be a proper map such that F_0 and F_1 are Z -embeddings (i.e. their images are Z -sets in M). Then there exists an isotopy $H: M \times I \rightarrow M$ such that $H_0 = \text{id}_M$ and $H_1 F_0 = F_1$. Moreover, if F is limited by an open cover $\mathcal{U}$, then*

we may choose H to be also limited by $\mathcal{U}$ (this will be very useful when verifying the hypothesis of Bing's shrinking criterion).

LEMMA 1.1 (WELL KNOWN). Suppose we have a commutative diagram of metric spaces:

$$\begin{array}{ccc} & X & \\ f \swarrow & & \searrow h \\ Y & \xrightarrow{g} & Z \end{array}$$

then if f and h are near-homeomorphisms, g is also one.

PROOF (SKETCH). If F and H are homeomorphisms sufficiently near to f and h , respectively, HF^{-1} will be the desired approximation to g . $\square$

The following theorem is due to J. West; we sketch the simple proof of it given by R. D. Edwards; see [5], [14].

THEOREM 1.2. Let $\pi: M \rightarrow X$ be a CE map of a Q -manifold M onto an ANR X . Suppose that the union of the nondegenerate² point preimages of π lies in a Z -subset of M ; then π is a near-homeomorphism—or, equivalently, the decomposition $\{\pi^{-1}(x), x \in X\}$ of M is shrinkable.

PROOF (SKETCH). We verify the Bing shrinking criterion. Let A be a Z -subset of M which contains the union of the nondegenerate point preimages of π . Using the well-known fact that a CE map between ANR's is a fine homotopy equivalence (see appendix of [11], or [7] or [9]), we can find a homotopy f_t of $f_0 = \text{id}_M$ such that:

f_1 factorizes through π , i.e. there exists $\bar{f}_1: X \rightarrow M$ such that $f_1 = \bar{f}_1\pi$;

πf_t is as close to π as we want.

We can approximate $f_1|A$ by a very close Z -embedding. If we unknot this embedding, we have just obtained a homeomorphism which shrinks the decomposition $\{\pi^{-1}(x)|x \in X\}$. $\square$

The following special case of 1.2 may be called West's second mapping cylinder theorem.

COROLLARY 1.3. If $f: M \rightarrow X$ is a CE map of a Q -manifold M onto an ANR X , then $Z(f)$ (= mapping cylinder of f) is a Q -manifold and the quotient map $q: [0, 1] \times M \rightarrow Z(f)$ is a near homeomorphism.

PROOF. We remark that $Z(f)$ is an ANR (Theorem 1.2, p. 178 of [8]). Then we apply 1.2 to the CE map $q: [0, 1] \times M \rightarrow Z(f)$. $\square$

REMARK. 1.4. In the situation of 1.3, X is a Z -subset of the Q -manifold $Z(f)$; this follows by observing that X is locally negligible in $Z(f)$, whenever f is CE; see [11, §§2, 3, 6] for more details.

²"Nondegenerate" means "not a point".

2. The CE mapping theorem. The CE mapping theorem will follow easily from one special case of it, which we prove now.

LEMMA 2.1. *If $f: M \rightarrow N$ is a CE map between Q -manifolds, then the natural retraction $r: Z(f) \rightarrow N$ is a near-homeomorphism.*

PROOF. By 1.3 $Z(f)$ is a Q -manifold. We begin by inspecting the following commutative square

$$(*) \quad \begin{array}{ccc} Z(f) \cup_N N \times [0, 1] & \xrightarrow{\bar{r}} & N \times [0, 1] \\ \downarrow \bar{p} & & \downarrow p = \text{projection on } N \\ Z(f) & \xrightarrow{r} & N \end{array}$$

where $Z(f) \cup_N N \times [0, 1]$ is obtained by adding to $Z(f)$ a collar along N (see Figure 1), and $\bar{p}$ and $\bar{r}$ are the natural (identity) extension of p and r .

FIGURE 1

Lemma 1.1 shows that it suffices to prove that $p, \bar{p}$ and $\bar{r}$ are near-homeomorphisms.

To prove that $\bar{p}$ is a near-homeomorphism, we just remark that N is collared in $Z(f)$, because it is a Z -subset (see 1.4 and the collaring theorem p. 33 of [2]).

The fact that p is a near-homeomorphism is exactly the stability property of Q -manifolds.

We prove that $\bar{r}$ is a near-homeomorphism by verification of the Bing shrinking criterion.

If $\mathcal{V}$ is an open covering of $Z(f) \cup_N N \times [0, 1]$, and $\mathcal{U}$ is a covering of $N \times [0, 1]$, we must find a homeomorphism h of $Z(f) \cup_N N \times [0, 1]$ such that:

- (i) If $\mathcal{D} = \{\bar{r}^{-1}(n) | n \in N \times [0, 1]\}$, then $h(\mathcal{D})$ refines $\mathcal{V}$.
- (ii) $\bar{r}h$ and f are $\mathcal{U}$ -close.

We will construct h as a composition $h = h_1 h_2$.

First let $\mathcal{W}$ be an open cover of $N \times [0, 1]$ such that $\text{St } \mathcal{W}$ refines $\mathcal{U}$.

Applying Theorem 1.2 we construct h_1 a self-homeomorphism of $Z(f) \cup_N N \times [0, 1]$ such that (see Figure 2):

(i₁) $\forall x \in N$, $f^{-1}(x) (\subset M \subset Z(f))$ is taken by h_1 to a subset of some $V_x \in \mathcal{V}$.

(ii₁) $\bar{r} \circ h_1$ and $\bar{r}$ are $\mathcal{U}$ -close.

FIGURE 2

Consider now the open covering $h_1^{-1}(\mathcal{V})$ of $Z(f) \cup_N N \times [0, 1]$. Since for each $x \in N$ there exist some $V_x \in \mathcal{V}$ such that $f^{-1}(x) \subset h_1^{-1}(V_x)$, we can construct, using first a small collar of N in $Z(f)$, then the lines of the mapping cylinder, a self-homeomorphism h_2 of $Z(f) \cup_N N \times [0, 1]$, such that (see Figure 3):

(i₂) $\forall x \in N \exists V_x \in \mathcal{V} \ h_2(r^{-1}(x)) \subset h_1^{-1}(V_x)$.

(ii₂) $\bar{r}h_2$ and $\bar{r}$ are $\mathcal{U}$ -close.

FIGURE 3

Since the nondegenerate elements of $\mathcal{D}$ are precisely $\{r^{-1}(x) | x \in N\}$, it follows from (i₁) and (i₂) that $h(\mathcal{D})$ refines $\mathcal{V}$. Conditions (ii₁) and (ii₂) imply that $\bar{r}h$ and $\bar{r}$ are $\text{St}\mathcal{U}$ -close, and hence $\mathcal{U}$ -close by choice of $\mathcal{U}$. $\square$

We can now prove the CE mapping theorem.

THEOREM 2.2 [3], [4]. *A CE map $f: M \rightarrow N$ between Q -manifolds is a near-homeomorphism.*

PROOF. The theorem follows from the following commutative diagram and Lemma 1.1

$$\begin{array}{ccc}
 [0, 1] \times M & \xrightarrow{q} & Z(f) \\
 p \downarrow & & \downarrow r \\
 M & \xrightarrow{f} & N
 \end{array}$$

by noting that p , q and r are near-homeomorphisms (stability, Corollary 1.3 and Lemma 2.1, respectively). $\square$

3. The mapping cylinder theorem [12].

THEOREM [12]. *Let $f: Q \rightarrow M$ be a map from Q to a Q -manifold M . Then the mapping cylinder $Z(f)$ of f is a Q -manifold factor (i.e. $Z(f) \times Q$ is a Q -manifold).*

The proof of this theorem occupies the rest of this section.

In the next lemma we reduce the theorem to the problem of finding CE maps with arbitrarily small point preimages.

LEMMA 3.1 [1]. *Let X be an ANR which is a cell like image of a Q -manifold. Suppose that for each open cover $\mathcal{U}$ of X , there exists a CE map $f: X \rightarrow M$, where M is a Q -manifold, such that $\{f^{-1}(m) | m \in M\}$ refines $\mathcal{U}$; then X is a Q -manifold.*

PROOF. Let $\pi: N \rightarrow X$ be a CE map where N is a Q -manifold. We will verify the Bing shrinking criterion for the decomposition of $\{\pi^{-1}(x)\}$ of N given by π ; in particular, we obtain that X is homeomorphic to N , and hence is a Q -manifold.

We want to show that given any open cover $\mathcal{U}$ of X and any open cover $\mathcal{V}$ of N , there exists a self-homeomorphism h of N such that:

$$\{h(\pi^{-1}(x)) | x \in X\} \text{ refines } \mathcal{V};$$

πh and π are $\mathcal{U}$ -close.

To construct h we proceed as follows.

First let $f: X \rightarrow M$ be a CE map onto a Q -manifold such that $\{f^{-1}(m) | m \in M\}$ refines $\mathcal{U}$; using this property and the properness of f , we can find an open cover $\mathcal{W}$ of M such that $f^{-1}(\mathcal{W})$ refines $\mathcal{U}$. Now since $f\pi$ is a CE map between Q -manifolds, it is a near-homeomorphism, so we can find a self-homeomorphism h of N such that:

$$\{h((f\pi)^{-1}(m)) | m \in M\} \text{ refines } \mathcal{V};$$

$f\pi h$ and $f\pi$ are $\mathcal{W}$ -close.

One can easily verify that h is the desired homeomorphism. $\square$

We now return to the proof of the mapping cylinder theorem. We verify the hypothesis of Lemma 3.1 when $X = Z(f) \times Q$.

We first show that $Z(f)$ is a CE image of a Q -manifold. Let $\tilde{f}: Q \rightarrow M \times Q \times Q$ be defined by $\tilde{f}(q) = (f(q), q, 0)$ for each q of Q . One can easily define a CE map $\theta: Z(f) \rightarrow Z(f)$ in the following way:

$$\theta([t, q]) = [t, q] \in [0, 1] \times Q \cup_f M,$$

if $[t, q]$ belongs to the image of $Q \times Q$ in $Z(\bar{f})$.

$$\theta(m, q_1, q_2) = m \in M \subset Z(f), \text{ if } (m, q_1, q_2) \text{ is in } M \times Q \times Q.$$

Now since $\bar{f}$ is a Z -embedding, a collar of $\bar{f}(Q)$ in $M \times Q \times Q$ shows that $Z(\bar{f})$ is a Q -manifold.

We must now construct a CE map $\psi: Z(f) \times Q \rightarrow Y (= Q\text{-manifold})$ with small point preimages. By producing the map with the identity on a large number of coordinates of Q , it suffices, in fact, to construct a CE map $\psi: Z(f) \times Q \rightarrow Y$ with point preimages small in the direction of $Z(f)$ (i.e. the projection of point preimages on $Z(f)$ are small). Since Q is homeomorphic to $Q \times Q$, we can replace $Z(f) \times Q$ by $Z(f) \times Q \times Q$. We take for Y the subset of $Z(f) \times Q \times Q$ defined by $Y = M \times Q \times Y \cup \text{Im}(I \times \Delta_Q \times 0)$, where $I \times \Delta_Q \times 0 = \{(t, q, q, 0) \in I \times Q \times Q \times Q\}$ and $\text{Im}(I \times \Delta_Q \times 0)$ is the image of this set in $Z(f) \times Q \times Q$. Observe that Y is homeomorphic to $Z(\bar{f})$, and hence is a Q -manifold.

The existence of ψ follows easily from the following lemma:

LEMMA 3.2. *There exists a CE map*

$$\bar{\psi}: I \times Q \times Q \times Q \rightarrow I \times \Delta_Q \times 0 \cup 1 \times Q \times Q \times Q$$

such that:

- (i) $\bar{\psi}|_{1 \times Q \times Q \times Q}$ is the identity.
- (ii) $\bar{\psi}$ has small point preimages in the $I \times Q$ direction.

PROOF. We sketch here an elementary proof of this lemma, which can be proved in many ways.

Fix $\epsilon \in (0, 1)$; then construct α and β maps from $[0, 1]$ onto itself such that the graphs look like

GRAPH α

GRAPH β

Then we define $\bar{\psi}$ by

$$\bar{\psi}(t, q, q', q'') = (\alpha(t), q, \beta(t)q' + (1 - \beta(t))q, \beta(t)q'').$$

If ϵ is sufficiently small, $\bar{\psi}$ is the map we are seeking. $\square$

REFERENCES

1. Z. Cerin, *On cellular approximations of manifolds*, 1974 (preprint).
2. T. A. Chapman, *Notes on Hilbert cube manifolds*, Univ. of Kentucky, 1972 (preprint).
3. ———, *Cell like mappings of Hilbert cube manifolds: solution of a handle problem*, Gen. Topology and Appl. **5** (1975), 123–145.
4. ———, *Cell like mappings of Hilbert cube manifolds: applications to simple homotopy theory*, Bull. Amer. Math. Soc. **79** (1973), 1286–1291.
5. R. D. Edwards, *Siebenmann's variation of West's proof of the ANR theorem* (manuscript).
6. M. Handel, *The Bing staircase construction for Q -manifolds*, 1975 (preprint).
7. W. Haver, *Cell like mappings between ANR's are fine homotopy equivalences*, Pacific J. Math. **58** (1975), 457–461.
8. S. T. Hu, *Theory of retracts*, Wayne State Univ. Press, Detroit, Mich., 1965.
10. A. Marin and Y. M. Visetti, *A general proof of Bing shrinkability criterion*, Proc. Amer. Math. Soc. **53** (1975), 501.
11. H. Toruńczyk, *Concerning locally homotopy negligible sets and characterization of l^2 -manifolds*, Fund. Math. (to appear).
12. J. E. West, *Mapping cylinder of Hilbert cube factors*, General Topology and Appl. **1** (1971), 111–125.
13. R. C. Lacher, *Cell-like mappings. I*, Pacific J. Math. **30** (1969), 717–731.
14. T. A. Chapman, *Lectures on Hilbert cube manifolds*, delivered at Guilford College, (October 1975), 1976 (preprint).

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