A NORMAL FORM THEOREM FOR LATTICES COMPLETELY GENERATED BY A SUBSET

GEORGE GRÄTZER AND DAVID KELLY¹

ABSTRACT. For an m -complete lattice L (m is an infinite regular cardinal) and subset X of L that m -generates L , we prove a Normal Form Theorem for elements of L expressed as polynomials over X . This generalizes a theorem of B. Jónsson in which such a representation is found for the lattice L freely m -generated by a poset X . We also apply this result to free m -products of m -complete lattices.

Throughout this paper, m is an infinite regular cardinal. A lattice L is *m-complete* (or L is an *m-lattice*) iff for any nonempty $S \subseteq L$ with $|S| < m$, the join and meet of S exist in L .² An m -lattice L is *m-generated* by a set X iff $L = X_m$, where the subsets X_α of L ($\alpha \leq m$) are inductively defined by: $X_0 = X$; for $\alpha > 0$, X_α consists of all elements of the form $\bigvee S$ or $\bigwedge S$ where $S \subseteq \bigcup (X_\beta \mid \beta < \alpha)$ and $0 < |S| < m$. (Equivalently, L is the smallest m -sublattice of L that contains X .)

Henceforth, L and X are as above. The *rank* of $a \in L$ (with respect to X), denoted by $\rho(a)$, is the least α such that $a \in X_\alpha$. Whenever we write $\bigvee T$, it is to be understood that $T \subseteq L$ and $0 < |T| < m$. An element $a \in L$ is *strictly m-join-reducible* (with respect to X) iff $a = \bigvee T$ for some $T \subseteq L$ and $\rho(t) < \rho(a)$ for all $t \in T$; in this case, the representation $a = \bigvee T$ is called *strict*. Any element that is not strictly m -join-reducible is called *strictly m-join-irreducible*. We shall also use the dual concepts.

Every element of X is strictly m -join-irreducible and strictly m -meet-irreducible. Obviously, every element of $L - X$ is strictly m -join-reducible or strictly m -meet-reducible. Our main result is the

NORMAL FORM THEOREM. *Let L be an m -lattice that is m -generated by the set X . If $a \in L$ is strictly m -join-reducible, then there is a strict representation $a = \bigvee T$ such that*

- (i) *every element of T is strictly m -join-irreducible;*
- (ii) *if $t = \bigwedge S$ is a strict representation of some $t \in T$, then $s \not\leq a$ for all $s \in S$.*

Received by the editors November 17, 1976.

AMS (MOS) subject classifications (1970). Primary 06A23.

Key words and phrases. Complete lattice, representation, join-irreducible.

¹The research of both authors was supported by the National Research Council of Canada.

²Our usage of the term " m -complete" is different from the customary usage in the literature.

© American Mathematical Society 1978

1. Applications. Before coming to the proof of the Normal Form Theorem, we will give two of its corollaries. An element a of an m -lattice L is *m-join-reducible* iff it has a representation $a = \bigvee T$ with $a \notin T$. The first corollary, due to B. Jónsson [3], motivated the present paper.

COROLLARY 1. *Let L be the completely free m -lattice generated by the poset X . If $a \in L$ is m -join-reducible, then there is a representation $a = \bigvee T$ such that*

- (i) *every element of T is m -join-irreducible;*
- (ii) *if $t \in T - X$, then there is a representation $t = \bigwedge S$ such that $s \not\leq a$ for all $s \in S$.*

PROOF. By P. Crawley and R. A. Dean [1], no element of L can be both m -join-reducible and m -meet-reducible, and every element of X is m -join-irreducible. Therefore, m -join-reducible and strictly m -join-reducible are equivalent properties in L .

COROLLARY 2. *Let L be a free m -product of the family $(L_i | i \in I)$ of m -complete lattices, and set $X = \bigcup (L_i | i \in I)$, a subset of L . If $a \in L - X$ is m -join-reducible, then there is a representation $a = \bigvee T$ such that*

- (i) *every element of $T - X$ is m -join-irreducible;*
- (ii) *if $t \in T - X$, then there is a representation $t = \bigwedge S$ such that $s \not\leq a$ for all $s \in S$;*
- (iii) *if³ $t \leq a_{(i)}$ for $t \in T$ and $i \in I$, then $t = a_{(i)}$.*

PROOF. By G. Grätzer and D. Kelly [2], no element of $L - X$ can be both m -join-reducible and m -meet-reducible. Therefore, m -join-reducible and strictly m -join-reducible are equivalent properties in $L - X$. Let $a = \bigvee T_0$ be a strict representation satisfying the conditions of the theorem. For each $t \in T_0$ that satisfies $t \leq a_{(i)}$ for some $i \in I$, replace t in T_0 by $a_{(i)}$, to form T . Then $a = \bigvee T$ and this is a representation satisfying the conditions of the corollary.

In [2], the second corollary is applied to show that the free m -product construction preserves semidistributivity. Similarly as in [3], we can prove a minimal property of the representation in Corollary 2. (We use the Structure Theorem of [2] in the proof.)

PROPOSITION. *Let L , X and $a = \bigvee T$ be as in Corollary 2. If $a = \bigvee T'$ is any representation, then for each $t \in T - X$ there exists $t' \in T'$ with $t \leq t'$.*

PROOF. Let $t \in T - X$ and $t = \bigwedge S$ be a representation that satisfies (ii) of Corollary 2, and consider the valid inequality $\bigwedge S \leq \bigvee T'$. By (iii),⁴ (C) cannot apply. Also, by (ii), $(\bigwedge W)$ cannot apply, leaving only $(W_\vee)$; this means that $t \leq t'$ for some $t' \in T'$.

The reader may find it of interest to compare Corollary 2 with the minimal

³ $a_{(i)}$ is the lower cover of a in L_i , see [2].

⁴(C), $(\bigwedge W)$, $(W_\vee)$ refer to conditions in the Structure Theorem of [2].

form theorem for free products in the finitary case, see H. Lakser [4].

2. Proof of the theorem. It is easily shown by induction on the rank that for any strictly m-join-reducible element a of L , there is $T \subseteq L$ such that $a = \bigvee T$ is a strict representation and (i) holds. From now on, a is a fixed strictly m-join-reducible element of L .

Following B. Jónsson [3], for $R \subseteq L$, $C(R)$ denotes the set of elements $b \in L$ such that $b \in R$ or $b \leq \bigvee T$, where $T \subseteq R$ and $\rho(t) < \rho(b)$ for all $t \in T$. For $R, R' \subseteq L$, $R \prec R'$ will mean that $R \subseteq C(R')$. Since $C(C(R)) = C(R)$, the relation $\prec$ quasi-orders the class of all subsets of L .

LEMMA. *Let $\mathcal{C}$ be a chain (with respect to inclusion) of subsets of L whose intersection is D . If $R \prec S$ whenever $R \supseteq S$ in $\mathcal{C}$, then $R \prec D$ for every $R \in \mathcal{C}$.*

PROOF. Let $b \in R$ and $R \in \mathcal{C}$. We show that $b \in C(D)$ by induction on $\rho(b)$. We can assume that $b \notin D$. Thus, $b \notin S$ for some $S \in \mathcal{C}$. Therefore, $R \supset S$; this implies that $R \prec S$. Hence, $b \in C(S)$ and $b \leq \bigvee T$ for some $T \subseteq S$ such that $\rho(t) < \rho(b)$ for all $t \in T$. By induction, $T \subseteq C(D)$, and therefore, $b \in C(C(D)) = C(D)$.

We now return to the proof of the theorem. Let A be the set of elements b of L such that $\rho(b) < \rho(a)$, $b \leq a$, and b is strictly m-join-irreducible. Let $\mathcal{K}$ be the class of all subsets R of L such that $R \prec R'$ implies that $R = R'$ whenever $R' \subseteq R$.

We define a sequence of subsets of A . Let $R_0 = A$. If R_α is defined and $R_\alpha \in \mathcal{K}$, we terminate the sequence and set $R = R_\alpha$; if $R_\alpha \notin \mathcal{K}$, then choose $b \in R_\alpha$ such that $R_\alpha \prec (R_\alpha - \{b\})$ and set $R_{\alpha+1} = R_\alpha - \{b\}$. If the R_β have been defined for $\beta < \alpha$, where α is a limit ordinal, then we define $R_\alpha = \bigcap (R_\beta \mid \beta < \alpha)$. By the Lemma, $A \prec R_\alpha$ whenever R_α is defined. Since the sequence is strictly decreasing, it must terminate; thus $R \in \mathcal{K}$ is defined and $A \prec R$.

It is clear that R satisfies condition (i) of the theorem. Assuming that (ii) fails, there is $r \in R$ and a strict representation $r = \bigwedge S$ such that $s \leq a$ for some $s \in S$. If s is strictly m-join-reducible, let $s = \bigvee B$ be a strict representation such that (i) holds; otherwise, $B = \{s\}$. In either case, $B \subseteq A$. Thus, $B \prec R$ and since $\rho(b) < \rho(r)$ for all $b \in B$, we conclude that $B \prec (R - \{r\})$. It follows that $R \prec (R - \{r\})$, which contradicts $R \in \mathcal{K}$, concluding the proof that R satisfies (ii).

We have already observed that there is $A_0 \subseteq A$ with $0 < |A_0| < m$ such that $a = \bigvee A_0$. Thus there is $T \subseteq R$ with $0 < |T| < m$ such that $A_0 \prec T$. Clearly, $a = \bigvee T$ and this representation satisfies all the conditions of the theorem.

REFERENCES

1. P. Crawley and R. A. Dean, *Free lattices with infinite operations*, Trans. Amer. Math. Soc. **92** (1959), 35-47.

2. G. Grätzer and D. Kelly, *Free m-products of lattices* (to appear). See also Notices Amer. Math. Soc. **24** (1977), A-1, A-221, A-287.
3. B. Jónsson, *Arithmetic properties of freely α -generated lattices*, Canad. J. Math. **14** (1962), 476–481.
4. H. Lakser, *Normal and canonical representations in free products of lattices*, Canad. J. Math. **2** (1970), 394–402.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG, MANITOBA, CANADA
R3T 2N2