**MAPPINGS OF THE INTERVAL
WITH FINITELY MANY PERIODIC POINTS
HAVE ZERO ENTROPY**

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ABSTRACT. It is shown that if f is a continuous map of a closed interval into itself, and f has finitely many periodic points, then the topological entropy of f is zero.

1. Introduction. Let I denote a closed interval and $C^0(I, I)$ denote the space of continuous functions of I into itself. For $f \in C^0(I, I)$ let $\text{ent}(f)$ denote the topological entropy of f (see [1], [2], [4], or [6] for a definition). The main result of this paper is the following.

THEOREM A. *Let $f \in C^0(I, I)$ and suppose that f has finitely many periodic points. Then $\text{ent}(f) = 0$.*

To prove Theorem A we prove the following theorem. Let $\Omega(f)$ denote the nonwandering set (see §2 for definitions).

THEOREM B. *Let $f \in C^0(I, I)$. Suppose f has finitely many periodic points, and all periodic points of f are fixed points of f . Then $\Omega(f)$ is the set of fixed points of f .*

Theorem A follows easily from Theorem B, using the following two elementary facts (see §3 for details).

LEMMA 1. *(proved by Adler, Konheim and McAndrew in [1]). Let f be a continuous map of a compact topological space and let n be a positive integer. Then $\text{ent}(f^n) = n \cdot \text{ent}(f)$.*

LEMMA 2 *(proved by Bowen [6]). Let f be a continuous map of a compact metric space and suppose $\Omega(f)$ is finite. Then $\text{ent}(f) = 0$.*

We remark that this paper uses some results from [3] which will be stated in §2. The main result of [3], which is somewhat related to Theorem B, is the following.

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THEOREM C. *Let $f \in C^0(I, I)$ and suppose $\Omega(f)$ is finite. Then $\Omega(f)$ is the set of periodic points of f .*

We conclude this section by noting that Theorems B and C leave open the following question.

If $f \in C^0(I, I)$ and f has finitely many periodic points, is $\Omega(f)$ the set of periodic points of f ?

2. Preliminary definitions and results. Let $f \in C^0(I, I)$. Let f^0 denote the identity map. For any positive integer n , we define f^n inductively by $f^1 = f$ and $f^n = f \circ f^{n-1}$. A point $x \in I$ is said to be periodic if $f^n(x) = x$, for some positive integer n . A point $x \in I$ is said to be eventually periodic, if $f^n(x)$ is periodic for some positive integer n . We say $x \in I$ is wandering if for some neighborhood W of x , $f^n(W) \cap W = \emptyset$ for every positive integer n . The set of points which are not wandering is called the nonwandering set and is denoted $\Omega(f)$.

Now, let $f \in C^0(I, I)$ and let p be a fixed point of f . We define the unstable manifold $W^u(p, f)$ and one-sided unstable manifold $W^u(p, f, +)$ as follows. Let $x \in W^u(p, f)$ if for any neighborhood V of x , $x \in f^n(V)$ for some positive integer n . Let $x \in W^u(p, f, +)$ if for any interval K with left endpoint p , $x \in f^n(K)$ for some positive integer n . It is easy to verify that $W^u(p, f)$ and $W^u(p, f, +)$ are connected.

We will use the following lemma, which is proved in §2 of [3]:

LEMMA 3. *Let $f \in C^0(I, I)$ and let p be a fixed point of f . Let $\overline{W^u(p, f)}$ denote the closure of $W^u(p, f)$. Then any element of $\overline{W^u(p, f)} - W^u(p, f)$ is periodic.*

We will also use the following theorem from [3] (see Theorem 7 of §3 of [3]):

THEOREM 4. *Let $f \in C^0(I, I)$. Suppose f has finitely many periodic points and p is a fixed point of f . If $x \in W^u(p, f)$ and $f(x) = p$, then $x = p$.*

Since $f(W^u(p, f)) \subset W^u(p, f)$ the following corollary follows immediately from Theorem 4.

COROLLARY 5. *Let $f \in C^0(I, I)$. Suppose f has finitely many periodic points, and p is a fixed point of f . If $x \in W^u(p, f)$ and $f^m(x) = p$ for some positive integer m , then $x = p$.*

We conclude this section by proving the following lemma.

LEMMA 6. *Let $f \in C^0(I, I)$. Suppose that f has finitely many periodic points, and all periodic points of f are fixed points of f . Let F denote the union of the set of fixed points of f and the set of endpoints of I . Let p and q be adjacent elements of F with $p < q$ (i.e. $p \in F$, $q \in F$, and $(p, q) \cap F = \emptyset$). Suppose for some $y \in (p, q)$, $f(y) < y$. Then for any $x \in (p, q)$ and any positive integer n , $f^n(x) < x$.*

PROOF. Since $f(y) < y$ for some $y \in (p, q)$ and (p, q) has no fixed points, $f(x) < x \ \forall x \in (p, q)$. Also, it follows that $f(p) = p$. This is true because if $f(p) \neq p$, then p is the left endpoint of I and $f(p) > p$. But $f(p) > p$ and $f(y) < y$ imply the existence of a fixed point in (p, y) , a contradiction.

Let n be a positive integer. First, suppose q is a fixed point of f . Since $f(p, q) \supset (p, q)$, there is a point $c \in (p, q)$ with $f^r(c) \in (p, q)$ for $r = 1, \dots, n$. Hence $f^n(c) < c$. Since f^n has no fixed points in (p, q) , $f^n(x) < x \ \forall x \in (p, q)$.

Now, suppose that q is not a fixed point of f . Then q is the right endpoint of I and $f^n(q) < q$. Again, since f^n has no fixed points in (p, q) , $f^n(x) < x \ \forall x \in (p, q)$. Q.E.D.

3. Proofs of Theorems A and B.

THEOREM B. Let $f \in C^0(I, I)$. Suppose f has finitely many periodic points, and all periodic points of f are fixed points of f . Then $\Omega(f)$ is the set of fixed points of f .

PROOF. Let $y \in I$ be any point which is not fixed. We will show that y is wandering.

Let F denote the union of the set of fixed points of f and the set of endpoints of I . There is a unique pair of elements of F , p_0 and q_0 , such that $p_0 < q_0$, $(p_0, q_0) \cap F = \emptyset$, and $y \in [p_0, q_0]$. Since y is not a fixed point of f , either $f(y) < y$ or $f(y) > y$. Without loss of generality, we may assume that $f(y) < y$. This implies, by Lemma 6, that for any $x \in (p_0, q_0)$ and any positive integer n , $f^n(x) < x$. Hence $f(p_0) = p_0$. We separate the proof into two cases.

Case 1. y is eventually periodic.

Then for some fixed point e of f , and some positive integer m , $f^m(y) = e$. Note that by Lemma 6, $e < y$. By Corollary 5, $y \notin W^u(e, f)$. Hence, by Lemma 3, $y \notin \overline{W^u(e, f)}$, since y is not periodic. Thus, for some $a \in I$ with $e < a < y$, $a \notin W^u(e, f)$.

Since $a \notin W^u(e, f)$, there is an open interval N containing e such that $a \notin \bigcup_{n \geq 0} f^n(N)$. Let $W = (b, c)$ be an open interval with $a < b < y < c$, and $f^m(W) \subset N$, and $f^r(W) \cap W = \emptyset$ for $r = 1, \dots, m$. Since $\bigcup_{n \geq 0} f^n(N)$ is a connected set containing e but not a , $W \cap (\bigcup_{n \geq 0} f^n(N)) = \emptyset$. Hence $f^r(W) \cap W = \emptyset$ for every positive integer r . Thus, y is wandering.

Case 2. y is not eventually periodic.

First suppose that for some positive integer n , $f^{n+1}(y) < f^n(y)$. This implies that $f^n(y)$ is not the left endpoint of I . Since $f^n(y) < y$, $f^n(y)$ is not the right endpoint of I . Thus, $f^n(y) \notin F$. Hence, there is a unique pair of elements of F , p_n and q_n , such that $p_n < q_n$, $(p_n, q_n) \cap F = \emptyset$ and $f^n(y) \in (p_n, q_n)$. By Lemma 5, for every positive integer m , $f^m(x) < x \ \forall x \in (p_n, q_n)$.

Let V be a neighborhood of y such that $f^r(V) \cap V = \emptyset$ for $r = 1, \dots, n$ and $f^n(V) \subset (p_n, q_n)$. Then $f^n(V)$ is to the left of V , and for every positive

integer m , $f^m(x) < x \forall x \in f^n(V)$. Hence $f^r(V) \cap V = \emptyset$ for every positive integer r . Thus, y is wandering.

Now, suppose that $f^{n+1}(y) \geq f^n(y)$ for every positive integer n . Since y is not eventually periodic, $f^{n+1}(y) > f^n(y)$ for every positive integer n . Let z be the limit of the increasing sequence $(f^n(y))$. Then $z > f^n(y)$ for every positive n , $z \leq p_0$ and $f(z) = z$.

We claim that $y \notin \overline{W^u(z, f, +)}$. Suppose that $y \in \overline{W^u(z, f, +)}$. Recall that for every positive integer m , $f^m(x) < x \forall x \in (p_0, q_0)$. Hence for any $x > p_0$, $x \notin \overline{W^u(p_0, f, +)}$. In particular, $y \notin \overline{W^u(p_0, f, +)}$. Hence $z \neq p_0$. Since $z \leq p_0$, this implies that $z < p_0$. Now since $\overline{W^u(z, f, +)}$ is connected, $[z, y] \subset \overline{W^u(z, f, +)}$. But $f(y) < z$ and $f(p_0) > z$. Hence, for some $x \in (p_0, y)$, $f(x) = z$. Since $x \in W^u(z, f, +)$, this is a contradiction, by Theorem 4. This establishes our claim that $y \notin \overline{W^u(z, f, +)}$.

Since $y \notin \overline{W^u(z, f, +)}$, there is an interval D with left endpoint z , such that $y \notin \bigcup_{n \geq 0} f^n(D)$. Let w be the greatest element of $\{x \in F: x < z\}$. Since z is the limit of the increasing sequence $(f^n(y))$, $f(x) > x \forall x \in (w, z)$, by Lemma 6. Hence there is an interval K with right endpoint z , such that $f(K) \subset K \cup D$.

For some positive integer j , $f^j(y)$ is in the interior of K . Let B be a neighborhood of y with $B \cap (\bigcup_{n \geq 0} f^n(D)) = \emptyset$ and $f^r(B) \cap B = \emptyset$ for $r = 1, \dots, j$, and $f^j(B) \subset K$. Then $f^r(B) \cap B = \emptyset$ for every positive integer r . Hence y is wandering. Q.E.D.

THEOREM A. Let $f \in C^0(I, I)$ and suppose f has finitely many periodic points. Then $\text{ent}(f) = 0$.

PROOF. Let n be the product of the periods of all periodic points of f . Then all periodic points of f^n are fixed points of f^n . By Theorem B, $\Omega(f^n)$ is the set of fixed points of f^n . In particular, $\Omega(f^n)$ is finite. Hence $\text{ent}(f^n) = 0$, by Lemma 2. Thus, by Lemma 1, $\text{ent}(f) = 0$. Q.E.D.

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