REMARKS ON A THEOREM OF KORANYI AND MALLIAVIN ON THE SIEGEL UPPER HALF PLANE OF RANK TWO

KENNETH D. JOHNSON

ABSTRACT. In [5], A. Koranyi and P. Malliavin showed that bounded functions on the Siegel upper half plane of rank two which satisfied two special elliptic differential equations were characterized by their values on the Bergman-Silov boundary. In this paper a simple proof of this theorem is given.

1. Introduction. The Siegel upper half plane of rank 2, H_2 is defined as the space of 2×2 -matrices Z with complex entries such that $Z = Z'$ ($'$ denotes transpose) and $\text{Im } Z > 0$ equipped with the metric

$$ds^2 = \text{tr} \left((Z - \bar{Z})^{-1} d\bar{Z} (Z - \bar{Z})^{-1} dZ \right).$$

The Laplace operator associated to this metric is easily seen to be

$$\Delta_1 = \text{tr} \left((Z - \bar{Z}) \partial_{\bar{Z}} (Z - \bar{Z}) \partial_Z \right)$$

when for $A = A'$, $\partial_A = (\partial_{ij})$ with $\partial_{ij} = \frac{1}{2}(1 + \delta_{ij})\partial/\partial a_{ij}$ and $\partial_{\bar{Z}}$ does not differentiate the entries of $Z - \bar{Z}$. Also, of interest to us is the operator

$$\Delta_2 = \text{tr} \left(\partial_{\bar{Z}} (Z - \bar{Z}) \partial_Z \right).$$

The Bergman-Silov boundary B_2 of H_2 contains the 2×2 real symmetric matrices S_2 as an open dense subset and the Poisson kernel of H_2 with respect to B_2 is the function $P_0: H_2 \times B_2 \rightarrow \mathbb{R}^+$ such that

$$P_0(Z, U) = c \left[\frac{\det(Z - \bar{Z})}{|\det(Z - U)|^2} \right]^{3/2} \quad (Z \in H_2, U \in S_2).$$

In [5], A. Koranyi and P. Malliavin showed that if $f \in L^\infty(H_2)$ and $\Delta_1 f = \Delta_2 f = 0$

$$f(Z) = \int_{S_2} P(Z, U) F(U) dU$$

for some $F \in L^\infty(S_2)$. In this paper, we give a new proof of this result which uses the techniques found in [4] rather than diffusion processes. We first introduce the group of symplectic transformations.

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If $Sp(2, \mathbf{R})$ is the group of 4×4 -matrices X with real entries which have the property that $XJ = JX'$ for

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \quad (I_2 \text{ is } 2 \times 2 \text{ identity matrix}),$$

$Sp(2, \mathbf{R})$ acts transitively on H_2 as follows. For $Z \in H_2$ and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbf{R})$ (A, B, C, D 2×2 -matrices) then

$$g(Z) = (AZ + B)(CZ + D)^{-1}.$$

Since $Sp(2, \mathbf{R})$ acts on H_2 it also acts on functions defined on H_2 . If $f: H_2 \rightarrow \mathbf{C}$,

$$Lgf(Z) = f(g^{-1}(Z)) \quad (g \in Sp(2, \mathbf{R})).$$

Then for $g \in Sp(2, \mathbf{R})$, $\Delta_1 Lgf = Lg\Delta_1 f$ and although $\Delta_2 Lgf \neq Lg\Delta_2 f$ in general it is true that if

$$g = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \quad (X = X') \quad \text{or} \quad g = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad (A \in O(2))$$

$\Delta_2 Lgf = Lg\Delta_2 f$ and if

$$g = \begin{pmatrix} aI_2 & 0 \\ 0 & a^{-1}I_2 \end{pmatrix} \quad (a > 0)$$

$Lg\Delta_2 f = a^{-2}\Delta_2 Lgf$. Thus, if $K(H_2) = \{f \in L^\infty(H_2): \Delta_1 f = \Delta_2 f = 0\}$, $Lg: K(H_2) \rightarrow K(H_2)$ for

$$g = \begin{pmatrix} aA & aAX \\ 0 & a^{-1}A \end{pmatrix} \quad (X = X', A \in O(2), a > 0).$$

Here we point out that Koranyi and Malliavin [5] in fact consider the operator $\Delta_2^1 = c(\text{tr}(Z - \bar{Z}))\Delta_2$ and then $Lg\Delta_2^1 f = \Delta_2^1 Lgf$ for g as above and $K(H_2) = \{f \in L^\infty(H_2): \Delta_1 f = \Delta_2^1 f = 0\}$.

If $f \in K(H_2)$ it follows from Furstenberg [1] that

$$f(Z) = \int_B P(Z, b)F(b)db$$

for some $F \in L^\infty(B)$ when B is the Furstenberg boundary of H_2 and $P: H_2 \times B \rightarrow \mathbf{R}^+$ a Poisson kernel and db is the $K = O(4) \cap Sp(2, \mathbf{R})$ -invariant measure on B normalized so that $\int_B db = 1$. After observing that $B = K/M$ where M is the diagonal matrices in K , we have that $P: G \times K \rightarrow \mathbf{R}^+$ where

$$P(g, k) = c \|g^{-1}k\vec{e}_3\|^{-2} \|g^{-1}k\vec{e}_3 \wedge \vec{e}_4\|^{-2}.$$

$\vec{e}_i$ is column vector with a 1 in the i th position 0's elsewhere and $\| \cdot \|$ is the norm on $\Lambda \mathbf{R}^n$.

2. Reduction to a special case. Suppose now that $f \in K(H_2)$ and

$$f(Z) = \int_B P(Z, b)F(b)db \quad (F \in L^\infty(B)).$$

Let K_0 be the group of all matrices g when

$$g = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad (A \in O(2))$$

and define for $\alpha \in C^\infty(K_0)$

$$\alpha *_{K_0} f(Z) = \int_{K_0} \alpha(k) f(k^{-1}(Z)) dk.$$

Then

$$\alpha *_{K_0} f(Z) = \int_B P(Z, b) (\alpha *_{K_0} F)(b) db.$$

Let $\hat{K}_0$ denote the set of equivalence classes of finite dimensional irreducible representations of K_0 and identify $\tau \in \hat{K}_0$ with its members. Then if we set $\chi_\tau(k) = (\deg \tau) \operatorname{tr} \tau(k)$ we have

(1) $\sum_{\tau \in \hat{K}_0} \chi_\tau *_{K_0} f$ converges to f in $C^\infty(H_2)$;

(2) $\sum_{\tau \in \hat{K}_0} \chi_\tau *_{K_0} F$ converges to F in $L^2(B)$.

Since $B_2 = K/\bar{K}_0$ to prove our result it suffices to show that each $\chi_\tau *_{K_0} F$ is a function on B_2 or S_2 .

Again let $\bar{N}$ be the group of all matrices g where

$$g = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \quad (X = X').$$

Select $\{\phi_n: n \geq 0\} \subset C_c^\infty(\bar{N})$ so that $\phi_n \rightarrow \delta$. Then $\phi_n *_{\bar{N}} f \rightarrow f$ as $n \rightarrow \infty$ in $C^\infty(H_2)$ and we see that

$$(\phi_n *_{\bar{N}} f)(Z) = \int_B P(Z, b) (\phi_n *_{\bar{N}} F)(b) db.$$

By observing that $\bar{N} \cong S_2$ and selecting our ϕ_n so that $\phi_n(AXA^{-1}) = \phi_n(X)$ for $A \in O(2)$, $X \in S_2$ we have that

$$\phi_n *_{\bar{N}} \chi_\tau *_{K_0} f = \chi_\tau *_{K_0} \phi_n *_{\bar{N}} f.$$

Writing $B = G/P$ for P some subgroup of G we have that $\bar{N}K_0P$ is open and dense in B and $B \sim \bar{N}K_0P$ has measure 0. Thus almost everywhere we may express F as an L^∞ function on $S_2 \times K_0 \cong S_2 \times S^1$

$$F(X; \theta) \quad (X \times e^{i\theta} \in S_2 \times S^1).$$

Convolving with a ϕ_n we have that if D is any differential operator with constant coefficients in the X -variables $\sup_{X, \theta} |DF(X, \theta)| < \infty$. After writing

$$X = \begin{pmatrix} x_1 + x_2 & x_3 \\ x_3 & x_1 - x_2 \end{pmatrix} \quad (x_1, x_2, x_3 \text{ real})$$

and setting $(x_2, x_3) = (r, 2\phi)$ (polar coordinates) we have that F is a function in the variables $(x_1, r, \phi; \theta)$. Convolving F over K_0 with some χ_τ we have that

$$F(x_1, r, 2\phi + 2\psi; \theta + \psi) = e^{2i\psi} F(x_1, r, \phi; \theta)$$

for some integer l and hence

$$F(x_1, r, 2\phi; \theta) = e^{2il\phi} F(x_1, r, 0, \theta - \phi).$$

Our result will follow if we can show that F is independent of θ .

3. **Main result.** For $Z \in H_2$ it will be convenient to use the coordinates

$$Z = \begin{pmatrix} z_1 + z_2 & z_3 \\ z_3 & z_1 - z_2 \end{pmatrix}$$

as was done in [5]. Then

$$\Delta_2^1 = c \left(y_1^2 \sum_{i=1}^3 \bar{\partial}_i \partial_i + \sum_{i=2}^3 y_1 y_i (\bar{\partial}_i \partial_1 + \bar{\partial}_1 \partial_i) \right) = {}^0\Delta_2 + \Delta_2^0$$

where

$${}^0\Delta_2 = \frac{c}{4} \left(y_1^2 \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} + \sum_{i=2}^3 2y_1 y_i \frac{\partial^2}{\partial x_1 \partial x_i} \right)$$

and

$$\Delta_2^0 = \frac{c}{4} \left(y_1^2 \sum_{i=1}^3 \frac{\partial^2}{\partial y_i^2} + \sum_{i=2}^3 2y_1 y_i \frac{\partial^2}{\partial y_1 \partial y_i} \right).$$

Suppose now that $f \in K(H_2)$ and

$$f(Z) = \int_{S^2 \times S^1} P(Z, X; \theta) F(X; \theta) dx d\theta$$

where $F(x_1; r, 2\phi; \theta) = e^{i\phi} F(x_1; r, 0; \theta - \phi)$ and $|\sup_{x, \theta} DF(X; \theta)| < \infty$ for D any differential operator in the x -variables with constant coefficients.

Let

$$H = \begin{pmatrix} -a_1 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix} \quad (a_1 > a_2 > 0)$$

and set $f_H(g) = \lim_{\tau \rightarrow \infty} f(g \exp \tau H)$. Then we have the following lemma.

LEMMA 3.1. $f_H = F$ almost everywhere and $({}^0\Delta_2 f)_H = 0$ on $S^2 \times S^1$.

PROOF. That $f_H = F$ almost everywhere is proved in Helgason and Koranyi [3]. That $({}^0\Delta_2 f)_H = 0$ follows first by differentiating under the integral to obtain

$$\frac{\partial^2}{\partial x_i \partial x_j} f(Z) = \int_{S^1} \int_{S^2} \frac{\partial^2}{\partial x_i \partial x_j} P(Z; U; \theta) F(U; \theta) dU d\theta$$

and by the conditions on F to obtain

$$\frac{\partial^2}{\partial x_i \partial x_j} f(Z) = \int_{S^1} \int_{S^2} P(Z; U; \theta) \frac{\partial^2}{\partial u_i \partial u_j} F(U; \theta) dU d\theta.$$

Our result now follows by combining the first statement of the lemma with the property that $y_i \rightarrow 0$ for each i .

We are now in a position to obtain our main result. To do this we must examine the equation $(\Delta_2^0 f)_H = 0$. A few more preliminaries are necessary to accomplish this.

If $Z = X + iY \in H_2$, $Z = g_x g_Y i$ for

$$g_x = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \quad \text{and} \quad g_Y = \begin{pmatrix} Y^{1/2} & 0 \\ 0 & Y^{-1/2} \end{pmatrix}$$

but $Z = g_x g_T i$ for some

$$g_T = \begin{pmatrix} T & 0 \\ 0 & T' \end{pmatrix} \quad \left(T = \begin{pmatrix} x & yz \\ 0 & z \end{pmatrix} \quad x, y, z \text{ real} \right).$$

An easy computation yields

$$\begin{aligned} y_1 &= \frac{x^2 + (y^2 + 1)z^2}{2}, & y_2 &= \frac{x^2 + (1 - y^2)x^2}{2}, & y_3 &= yz^2, \\ \frac{\partial}{\partial y_1} &= \frac{1 + y^2}{2x} \frac{\partial}{\partial x} - \frac{y}{z^2} \frac{\partial}{\partial y} + \frac{1}{2z} \frac{\partial}{\partial z}, \\ \frac{\partial}{\partial y_2} &= \frac{1 - y^2}{2x} \frac{\partial}{\partial x} + \frac{y}{z^2} \frac{\partial}{\partial y} - \frac{1}{2z} \frac{\partial}{\partial z}, \\ \frac{\partial}{\partial y_3} &= \frac{-y}{x} \frac{\partial}{\partial x} + \frac{1}{z^2} \frac{\partial}{\partial y}. \end{aligned}$$

Modifying the arguments of Lemma 2.9 of [4] to see that $(\partial f / \partial x)_H = (\partial f / \partial z)_H = 0$ we obtain

$$\text{LEMMA 3.2. } (\Delta_2^0 f)_H = cd / dy ((1 + y^2/2)^2 d / dy F(X; \tan^{-1}y)) = 0.$$

Thus we obtain setting $\theta = \tan^{-1}y$ that $d / d\theta (\sec^2 \theta d / d\theta) F(X; \theta) = 0$ and solving we have

$$F(x_1; r, 2\phi; \theta) = F_1(x_1; r, 2\phi) + F_2(x_1; r, 2\phi; 0)(\sin 2\theta + 2\theta)$$

when F_1 is independent of θ . Recall that

$$F(x_1; r, 2\phi; \theta) = e^{i\phi} F(x_1; r, 0; \theta - \phi).$$

Then

$$\begin{aligned} e^{i\phi} F(x_1; r, 0, \theta - \phi) \\ = e^{i\phi} (F_1(x_1; r, 0) + F_2(x_1; r, 0; 0)(\sin 2(\theta - \phi) + (\theta - \phi))). \end{aligned}$$

As $F(x_1; r, 2\phi; 0) = F_1(x_1; r, 2\phi)$ we have

$$F_1(x_1; r, 2\phi) = e^{i\phi} (F_1(x_1; r, 0) - F_2(x_1; r, 0; 0)(\sin 2\phi + 2\phi))$$

and so

$$\begin{aligned} e^{i\phi} (F_1(x_1; r, 0) + F_2(x_1; r, 0; 0)(\sin 2(\theta - \phi) + 2(\theta - \phi))) \\ = e^{i\phi} (F_1(x_1; r, 0) - F_2(x_1; r, 0; 0)(\sin 2\phi + 2\phi)) \\ + F_2(x_1; r, 2\phi; 0)(\sin 2\theta + 2\theta). \end{aligned}$$

Thus

$$\begin{aligned} F_2(x_1; r, 2\phi; 0)(\sin 2\theta + 2\theta) \\ = F_2(x_1; r, 0; 0)e^{i\phi} ((\sin 2\phi + 2\phi) + (\sin 2(\theta - \phi) + 2(\theta - \phi))) \end{aligned}$$

and dividing by $\sin 2\theta + 2\theta$ we see that the right-hand side of the equation is not independent of θ unless $F_2 = 0$ and finally we obtain the lemma.

LEMMA 3.3. *F is independent of θ .*

REMARK. That $F_2 = 0$ also follows immediately from the boundedness of F . Finally we summarize.

THEOREM 3.4 (KORANYI AND MALLIAVIN [5]). *If $f \in K(H_2)$*

$$f(Z) = \int_{B_2} P_0(Z, b)F(b)db$$

for some $F \in L^\infty(B_2)$.

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DEPARTMENT OF MATHEMATICS, SWAIN HALL-EAST, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401

Current address: Department of Mathematics, University of Georgia, Athens, Georgia 30602