REMARKS ON SUBSEQUENCES, SUBSERIES AND REARRANGEMENTS

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ABSTRACT. In this paper we shall evaluate the measure theoretic, as well as topological (Baire category theoretic) complexity of the set of convergent subsequences of a given sequence of real numbers. Similar problems regarding subseries and rearrangements of a given series are also discussed.

1. Subsequences of a given sequence. In what follows, 2^ω stands for the space of infinite sequences of zeroes and ones equipped with the product of discrete topologies on the one hand and product of $\frac{1}{2}$, $\frac{1}{2}$ measures on the other hand. $T: 2^\omega \rightarrow 2^\omega$ is the map $(T\varepsilon)_i = 1 - \varepsilon_i$. $\Omega \subset 2^\omega$ is the set of those sequences which have infinitely many zeroes and ones. Note that Ω^c is countable and, hence, Ω is a G_δ set and is of measure one, which is left invariant by T . Note also that T is a homeomorphism and is measure preserving.

A property defined for points of 2^ω is said to hold almost everywhere if the set of points for which the property holds is of cofirst category or is of measure one.

Let now $\mathbf{a} = (a_n)$ be a sequence of real numbers. For any ε in Ω , $\varepsilon\mathbf{a}$ is the subsequence of $\mathbf{a}$ given by (a_{n_k}) where $n_1 < n_2 < n_3 < \dots$, and $\varepsilon_i = 1$ if and only if $i = n_k$ for some k .

THEOREM 1. *Let α be limit point of $\mathbf{a}$. Then, $A_\alpha = \{\varepsilon: \alpha \text{ is a limit point of } \varepsilon\mathbf{a}\}$ is a cofirst category set and is of measure one.*

PROOF. $A_\alpha = \bigcap_p \bigcup_n \bigcap_{k>n} \{\varepsilon: \varepsilon_k = 1 \text{ and } |a_k - \alpha| < 1/p\}$, and, hence, is a Borel set. (We remark at this point that various sets we encounter subsequently can be proved to be Borel by similar arguments.) Further, the set A_α is a tail set and, hence, either A_α or $\Omega - A_\alpha$ is of first category [3, p. 84]. But since $T(\Omega - A_\alpha) \subset A_\alpha$, A_α cannot be of first category. The statement regarding measure follows by a similar argument using Kolmogorov's zero-one law [2, p. 201].

COROLLARY 1. *Let P be the set of all limit points of $\mathbf{a}$. Then $A = \{\varepsilon: P \text{ is the set of all limit points of } \varepsilon\mathbf{a}\}$ is a cofirst category set and is of measure one.*

PROOF. If $\alpha_1, \alpha_2, \dots$ is a countable subset of P , which is dense in P , then $A = \bigcap_i A_{\alpha_i}$.

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COROLLARY 2. *If $\limsup a = \beta$ and $\liminf a = \mu$ then, $B = \{\epsilon: \limsup \epsilon a = \beta \text{ and } \liminf \epsilon a = \mu\}$ is a cofirst category set and is of measure one.*

COROLLARY 3. *If a sequence is not convergent then almost every subsequence is not convergent, that is, if a is not convergent then $C = \{\epsilon: \epsilon a \text{ is not convergent}\}$ is a cofirst category set and is of measure one.*

2. Subseries of a given series. If $s = \sum a_n$ is a series of real numbers and ϵ is in Ω , then ϵs is the series $\sum \epsilon_i a_i$, or, equivalently, the subseries of $\sum a_n$ given by $\sum a_{n_k}$, where $n_1 < n_2 < \dots$ and $\epsilon_i = 1$ if and only if $i = n_k$ for some k .

THEOREM 2. (i) *If s diverges to $+\infty$ then, in both senses, almost all subseries diverge to $+\infty$.*

(ii) *If s diverges to $-\infty$ then, in both senses, almost all subseries diverge to $-\infty$.*

(iii) *If s oscillates then, in both senses, almost all subseries oscillate.*

Proof is similar to that of Theorem 1 and, hence, is omitted. This theorem gives us the following corollary which should be compared with [4] and Theorems 1.5 and 1.10 of [5].

COROLLARY 4. *If s is not convergent then, in both senses, almost all subseries are not convergent.*

The question that now arises is whether almost every subseries of a convergent series is convergent. A solution is given by the following theorem.

THEOREM 3. *Let $s = \sum a_n$ be a convergent series. Then either almost every subseries is convergent or almost every subseries is not convergent. In the measure theoretic sense the first or second case occurs according as $\sum a_n^2 < \infty$ or $\sum a_n^2 = \infty$. In the category theoretic sense the first or second case occurs according as $\sum |a_n| < \infty$ or $\sum |a_n| = \infty$.*

PROOF. The first statement is a consequence of the zero-one laws. The second statement can be derived from Kolmogorov's three series theorem [2, p. 199] as follows: Consider $X_n(\epsilon) = \epsilon_n a_n$. Since $\sum a_n$ is convergent, X_n is a uniformly bounded sequence of random variables with $\sum X_n$ convergent almost surely if and only if $\sum \text{Var}(X_n) = \sum E(X_n^2) - (EX_n)^2 < \infty$, that is, if and only if $\sum(a_n^2/2 - a_n^2/4) = \sum a_n^2/4 < \infty$.

The final statement of the theorem will be proved using the following lemma.

LEMMA. *Let X be a metric space and f_n be a sequence of real continuous functions on X .*

(i) *If $\{x: \sup_n f_n(x) < \infty\}$ is of second category then there is a nonempty open set U such that $\sup_n \sup_{x \in U} f_n(x) < \infty$.*

(ii) *If $\{x: \inf_n f_n(x) > -\infty\}$ is of second category then there is a nonempty open set U such that $\inf_n \inf_{x \in U} f_n(x) > -\infty$.*

(iii) If $\{x: \sup_n |f_n(x)| < \infty\}$ is of second category then there is a nonempty open set U such that $\sup_n \sup_{x \in U} |f_n(x)| < \infty$.

PROOF. To prove (i) observe that $B_k = \{x: \sup f_n(x) < k\}$ is a closed set and $\bigcup_{k \geq 1} B_k$ is of second category. Other parts are similarly proved.

Now to prove the last part of Theorem 3, for $n \geq 1$ define f_n on Ω by $f_n(\epsilon) = \sum_{i \leq n} \epsilon_i a_i$. If almost every subseries is convergent, that is, $\{\epsilon: \epsilon s \text{ is convergent}\}$ is of second category, then by (iii) of the above lemma there is a nonempty open set U such that $\sup_n \sup_{x \in U} |f_n(\epsilon)| = k < \infty$. So there is an ϵ_0 and an integer m such that for any ϵ with $\epsilon(i) = \epsilon_0(i)$ for $i < m$, we have $\sup |f_n(\epsilon)| < k$ showing that s is absolutely convergent. Conversely, if s is absolutely convergent, then, in fact, every subseries is convergent.

EXAMPLE. In the measure theoretic sense, almost every subseries of $\sum (-1)^n/n$ is convergent, whereas almost every subseries of $\sum (-1)^n/\sqrt{n}$ is not. However, in the category sense, for any of these series almost no subseries converges.

3. Rearrangements of series. For a series $s = \sum a_n$ and a permutation π of $N = 1, 2, \dots$, the series $\sum a_{\pi(n)}$, which is a rearrangement of s , is denoted by πs . If N^N is given the product of discrete topologies on N , then N^N —the set of permutations of N —regarded as a subset of N^N is a topologically complete separable metric group. N is neither locally compact, nor does there seem to exist any natural measure on it. If a series s is absolutely convergent, then every rearrangement of s is convergent. Conversely, if every rearrangement of s is convergent, then s is absolutely convergent. The following theorem strengthens this statement.

THEOREM 4. If $\{\pi: \pi s \text{ is convergent}\}$ is of second category then s is absolutely convergent.

PROOF. Let $s = \sum a_n$ be the given series. For $n \geq 1$ define f_n on N^N by

$$f_n(\pi) = \sum_{i \leq n} a_{\pi(i)}.$$

Each f_n is continuous, and by (iii) of the lemma of §2 we get a nonempty open set U such that

$$\sup_n \sup_{\pi \in U} |f_n(\pi)| = K < \infty.$$

From the topology of N , this means that there is a π_0 and an integer m such that for any permutation π with $\pi(i) = \pi_0(i)$ for all $i < m$ we have $\sup_n |f_n(\pi)| < K$. Hence s is absolutely convergent.

From the proof of the theorem we can, in fact, deduce

COROLLARY 5. (a) If $\{\pi: \text{partial sums of } \pi s \text{ are bounded above}\}$ is of second category then $\epsilon s = \sum \epsilon_n a_n < \infty$, where $\epsilon_n = 1$ or 0 according as a_n is positive or negative.

(b) If $\{\pi: \text{partial sums of } \pi s \text{ are bounded below}\}$ is of second category then

$\varepsilon s = \sum \varepsilon_n a_n < \infty$, where $\varepsilon_n = -1$ or 0 according as a_n is negative or positive.

As a consequence we have the following result of Agnew [1].

COROLLARY 6. *If s is conditionally convergent then almost every rearrangement is not convergent. More precisely, almost every rearrangement πs oscillates between $+\infty$ and $-\infty$.*

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