A NEW GENERALIZATION OF THE STURM COMPARISON THEOREM TO SELFADJOINT SYSTEMS

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ABSTRACT. The Sturm Comparison Theorem is generalized to second order linear systems. It is based on a comparison of the elements of the matrices involved.

In this paper, we consider the vector differential equations

$$(1) \quad y'' + P(t)y = 0$$

and

$$(2) \quad z'' + Q(t)z = 0,$$

where $P(t) = (p_{ij}(t))$ and $Q(t) = (q_{ij}(t))$ are continuous $n \times n$ symmetric matrices on a given interval $[a, b]$. For the case $n = 1$ such equations have been studied extensively, beginning with the work of Sturm [7] in 1836. Since then there have been various extensions of the Sturmian theory to selfadjoint systems of second order linear differential equations, initiated by Morse [5] in 1930. Further extensions were subsequently given by Birkhoff and Hestenes, Reid, and others (see [6]). It was shown in [5] that if $P(t)$ and $Q(t)$ are symmetric, $Q(t) \geq P(t)$, i.e. $Q(t) - P(t)$ is positive semidefinite, with $Q(\bar{t}) > P(\bar{t})$ for some number $\bar{t}$ in the interval $[a, b]$, and if (1) has a nontrivial solution $y(t)$ satisfying $y(a) = y(b) = 0$, then (2) has a nontrivial solution $z(t)$ such that $z(a) = z(c) = 0$, where c is some number in the open interval (a, b) . The purpose of this note is to present a similar theorem which is based on an elementwise comparison of the matrices $P(t)$ and $Q(t)$. Our theorem neither implies the theorem of Morse nor is it implied by it. We are able to give a relatively simple proof based on variational methods and an earlier result of ours.

We recall that a number b , $b > a$, is said to be *conjugate* to a relative to a certain equation of the form (1) if there exists a nontrivial solution of this equation which vanishes at a and b . The equation is said to be *disconjugate* on an interval I if no nontrivial solution of it vanishes more than once in I . It is well-known that in the selfadjoint case, i.e. if $P(t)$ is symmetric, conjugate points are isolated. Let $\mathcal{Q}[a, b]$ denote the set of absolutely continuous R^n -valued functions $h(t)$ on $[a, b]$ such that $|h'| \in L^2[a, b]$ and $h(a) = h(b)$.

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$= 0$. The proof of the following lemma, which essentially follows from Reid [6, p. 332], has been given in [2]. For more information on the preceding concepts one might consult [4] and [6].

LEMMA 1. Let $A(t)$ be a continuous $n \times n$ symmetric matrix on $[a, b]$, and let $J[h; a, b]$ denote the functional

$$J[h; a, b] = \int_a^b (\langle h', h' \rangle - \langle Ah, h \rangle) dt$$

over the set $\mathcal{Q}[a, b]$ of admissible functions. Then $J[h; a, b] \geq 0$ for all h in $\mathcal{Q}[a, b]$ if and only if the interval $[a, b]$ contains no point conjugate to a in its interior relative to the equation $x'' + A(t)x = 0$.

REMARK 1. The above lemma is a slight modification of the well-known fact (see, e.g., [3] or [6]) that the equation

$$(3) \quad x'' + A(t)x = 0$$

is disconjugate on the interval $[a, b]$ if and only if

$$J[h; a, b] = \int_a^b (\langle h', h' \rangle - \langle Ah, h \rangle) dt > 0$$

over the set of admissible functions h , $h \not\equiv 0$.

THEOREM 1. Assume that in equations (1) and (2), $q_{ij}(t) \geq p_{ij}(t)$ for $1 \leq i, j \leq n$, and $t \in [a, b]$. Further, assume that $q_{ii}(t) > p_{ii}(t)$ for some $i \in (a, b)$, $1 \leq i \leq n$, and that $p_{ij}(t) \geq 0$ for $i \neq j$. If (1) has a nontrivial solution $y(t)$ satisfying $y(a) = y(b) = 0$, then (2) has a nontrivial solution $z(t)$ such that $z(a) = z(c) = 0$, $a < c < b$.

We note that in the above theorem only the diagonal elements of Q are required to be strictly greater than those of P at a point i . It can be verified, by letting $P(t) = \text{diag}(0, 1)$ and $Q(t) = \text{diag}(1, 1)$, that this condition cannot be relaxed any further.

PROOF OF THEOREM 1. First we assume that b is the first conjugate point of a relative to (1). By Theorem 1 of [1] (or Theorem 2 of [2]) it follows that there exists a nontrivial solution

$$u(t) = \text{col}(u_1, \dots, u_n)$$

of (1) such that $u(a) = u(b) = 0$ and $u_j(t) \geq 0$ on $[a, b]$, $j = 1, \dots, n$. Let $J[h; a, b]$ and $\tilde{J}[h; a, b]$ define the functionals given by

$$J[h; a, b] = \int_a^b (\langle h', h' \rangle - \langle Ph, h \rangle) dt$$

and

$$\tilde{J}[h; a, b] = \int_a^b (\langle h', h' \rangle - \langle Qh, h \rangle) dt$$

over the set $\mathcal{Q}[a, b]$ of admissible functions. Then,

$$\begin{aligned}
\tilde{J}[u; a, b] &= \int_a^b (\langle u', u' \rangle - \langle Qu, u \rangle) dt \\
&= \int_a^b \left(\langle u', u' \rangle - \sum_{i=1}^n q_{ii}(t) u_i^2 - \sum_{i \neq j} q_{ij}(t) u_i u_j \right) dt \\
&< \int_a^b \left(\langle u', u' \rangle - \sum_{i=1}^n p_{ii}(t) u_i^2 - \sum_{i \neq j} p_{ij}(t) u_i u_j \right) dt \\
&= \int_a^b (\langle u', u' \rangle - \langle P(t)u, u \rangle) dt = J[u; a, b].
\end{aligned}$$

The above inequality follows from the hypothesis that $p_{ii}(\bar{t}) < q_{ii}(\bar{t})$, $1 \leq i \leq n$. Multiplying the equation

$$u'' + P(t)u = 0$$

by $-u$ and integrating from a to b we see that

$$\int_a^b (\langle u', u' \rangle - \langle Pu, u \rangle) dt = 0.$$

We have thus shown that $\tilde{J}[u; a, b] < J[u; a, b] = 0$. By Lemma 1 a has a conjugate point c in the open interval (a, b) relative to equation (2). Consequently, there exists a nontrivial solution $z(t)$ of (2) satisfying $z(a) = z(c) = 0$, where c is some number in the open interval (a, b) .

Now, let us assume that b is not the first conjugate point of a relative to (1). Let $\eta(a)$ be the first conjugate point of a , and let

$$v(t) = \text{col}(v_1, \dots, v_n)$$

be a solution of (1) such that $v(a) = v(\eta(a)) = 0$ and $v_j(t) \geq 0$, $j = 1, \dots, n$. Then $a < \eta(a) < b$. The same argument that we gave to establish that $\tilde{J}[u; a, b] < J[u, a, b]$ shows that

$$\tilde{J}[v; a, \eta(a)] \leq J[v; a, \eta(a)].$$

We no longer have strict inequality since we can not assume that $\bar{t} \in [a, \eta(a)]$. However, $\tilde{J}[v; a, \eta(a)] \leq J[v; a, \eta(a)] = 0$ implies (see Remark 1) that (2) is not disconjugate on $[a, \eta(a)]$. Hence the interval $[a, \eta(a)]$ contains a point conjugate to a relative to (2). For, it follows (see [3] or [6]) that if $\tilde{\eta}(a)$ is the first conjugate point of a relative to (2) then (2) is disconjugate on $[a, \tilde{\eta}(a)]$. Hence we must have $\tilde{\eta}(a) \leq \eta(a)$. This shows that (2) has a nontrivial solution $z(t)$ such that $z(a) = z(c) = 0$, where c is some number in the interval $(a, \eta(a)]$, and the proof is complete.

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