THE VOLUME OF A SLIGHTLY CURVED SUBMANIFOLD IN A CONVEX REGION

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ABSTRACT. Let T be a compact convex region in an n -dimensional Riemannian space, k_s be the minimum sectional curvature in T , and $\kappa > 0$ be the minimum normal curvature of the boundary of T . Denote by $P^r(\xi)$ a r -dimensional sphere, plane or hyperbolic plane of curvature ξ . We assume that k_s, κ are such that on $P^2(k_s)$ there exists a circumference of curvature κ . Let $R_0 = R_0(\kappa, k_s)$ be its radius. Now, let Q be a convex (in interior sense) m -dimensional surface in T whose normal curvatures with respect to any normal are not greater than χ satisfying $0 < \chi < \kappa$. Denote by L_χ the length of a circular arc of curvature χ in $P^2(k_s)$ with the distance $2R_0$ between its ends. We prove that the volume of Q does not exceed the volume of a ball in $P^m(k_s - (n-m)\chi^2)$ of radius $\frac{1}{2}L_\chi$. These volumes are equal when T is a ball in $P^n(k_s)$ and Q is its m -dimensional diameter.

1. Statement of the result. Let M be an n -dimensional Riemannian space, $n \geq 2$, of regularity class C^4 . Throughout this paper, it will be assumed that manifolds, their edges, surfaces, mappings etc. all are of class C^4 without a special mention. We consider in M a connected region that has a compact closure T and is bounded by a nonempty hypersurface Γ . We suppose that all the normal curvatures of Γ on the side of the interior normal are not less than some positive number κ . (Hence, Γ is two-sided and T is situated on one side of Γ .) Suppose that in the compact region T the sectional curvatures $\geq k_s > -\kappa^2$. In such a case we will call T *normal*.

Denote by $P^m(k_s)$ an m -dimensional sphere, Euclidean or hyperbolic space of curvature k_s . It was remarked in [3, §1], that on $P^2(k_s)$ there exists a circle T_0 whose circumference Γ_0 has geodesic curvature κ if and only if $k_s > -\kappa^2$.

Denote by Q a connected m -dimensional Riemannian manifold, $2 \leq m < n$, with an edge (possibly empty) $\partial Q \subset Q$. We call Q *convex* if for any $p, q \in Q, p \neq q$, there exists a minimal geodesic with the ends p, q .

An isometric immersion $i: Q \rightarrow T$ will be called χ -curved, $\chi \geq 0$, if the curvature of the image of any geodesic in Q under the mapping i does not exceed χ .

Let $\chi < \kappa$ and $i: Q \rightarrow T$ be a χ -curved immersion. It follows immediately from the corollary of Theorem 1 in [3, §1], that any geodesic in Q is not longer than the length L_χ of a circular arc in the circle T_0 which has geodesic

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curvature χ and whose ends are opposite points of Γ_0 . Therefore, in fact,

$$(1.1) \quad \partial Q \neq \emptyset.$$

If Q is convex, then obviously the diameter D of Q satisfies

$$(1.2) \quad D \leq L_\chi.$$

We prove the following

THEOREM. *Let a manifold Q be convex, $\chi < \kappa$ and $i: Q \rightarrow T$ be a χ -curved immersion. Then (i) Q is a compact manifold with a nonempty edge and (ii) volume of Q does not exceed the volume of a ball B of radius $\frac{1}{2} L_\chi$ in $P^m(k_s - (n - m)\chi^2)$.*

REMARKS. 1. The statement (i) has already been proved in the preceding remarks (see (1.1) and (1.2)).

2. Q and B have equal volumes when T is a ball in $M = P^n(k_s)$ and Q is an m -dimensional diameter of T . (Then $\chi = 0$.)

3. If $P^m(k_s - (n - m)\chi^2)$ for $m = 1$ is regarded as $R^1 = R$, then this theorem generalizes the corollary in [3, §1].

4. If Q is not convex, then its volume can be arbitrarily large.

5. If $\kappa \leq 0$ or $k_s \leq -\kappa^2$, then, in such a region, the submanifold Q can have arbitrarily large diameter and volume, even if the immersion i is 0-curved.

6. The smoothness requirement (C^4) is not really essential. It only allows for the use of results in [1], [2], [3] and could be reduced to C^2 by means of a suitable approximation.

7. It is easy to see that B is not more than a semisphere of $P^m(k_s - (n - m)\chi^2)$ when $k_s - (n - m)\chi^2 > 0$.

8. If $\frac{1}{2} L_\chi$ in this Theorem is replaced by L_χ , then such a rougher estimate can be obtained in a simpler way: instead of the remote point q defined in §2, we can use then an arbitrary point of Q .

2. Proof of Theorem. In view of Remark 1, we need to prove only (ii).

LEMMA 1. *Sectional curvatures of the manifold $Q \setminus \partial Q$ are not less than $k_s - (n - m)\chi^2$.*

PROOF. Let $q \in Q \setminus \partial Q$ and σ be a 2-dimensional direction at the point q . Denote by σ_* the image of σ under the mapping $i_{*q}: Q_q \rightarrow M_{i(q)}$. Let $k(\sigma)$ and $k(\sigma_*)$ be the curvatures of the manifolds Q and M^q in the directions σ and σ_* . From formula (9) in [4, point 3.7], we have that:

$$(2.1) \quad k(\sigma) = k(\sigma_*) + \sum_{\mu=1}^{n-m} \left| \begin{array}{cc} l_\mu(u, u) & l_\mu(u, v) \\ l_\mu(u, v) & l_\mu(v, v) \end{array} \right|$$

where $u, v \in \sigma$ are orthogonal unit vectors and $l_\mu(\cdot, \cdot)$ is the second quadratic form of Q with respect to a unit normal N_μ , $\mu = 1, 2, \dots, n - m$, satisfying $N_\mu \perp N_\nu$ when $\mu \neq \nu$.

Denote by u_μ, v_μ orthogonal unit vectors showing the principal directions of

the form l_μ restricted to σ . Then

$$(2.2) \quad \begin{vmatrix} l_\mu(u, u) & l_\mu(u, v) \\ l_\mu(u, v) & l_\mu(v, v) \end{vmatrix} = \begin{vmatrix} l_\mu(u_\mu, u_\mu) & 0 \\ 0 & l_\mu(v_\mu, v_\mu) \end{vmatrix} \geq -\chi^2.$$

It follows from the inequalities $k(\sigma_*) \geq k_s$ and (2.2) that $k(\sigma) \geq k_s - (n - m)\chi^2$. Q.E.D.

We will denote by $\rho(\cdot, \cdot)$ the intrinsic distance between the sets in the region T , and by ab we will denote a shortest path with the ends $a, b \in T$. We will also use the notation ab for $\rho(a, b)$. A shortest path ab always exists and it is a geodesic. This fact follows from an argument analogous to that contained in Lemma 2 and the remark after it in [1].

By compactness of Q , there exists a point $q \in Q$ such that $\rho(i(q), \Gamma) = \max_{p \in Q} \rho(i(p), \Gamma)$. We will call q the *remote* point of Q . Obviously, $i(q) \in \text{int } T$ because otherwise $i(Q) \subset \Gamma$ and the image under the immersion i of any geodesic in Q has the curvature not less than κ . This is obviously impossible by the definition of χ -curved immersion i , and by the fact that $\chi < \kappa$.

Let $b \in T$. A shortest path ba , $a \in \Gamma$, will be called *b-projecting* if $ba = \rho(b, \Gamma)$.

LEMMA 2. *Let q be a remote point of Q and let $b = i(q)$. Denote by $\Omega \subset Q_q$ the set of directions (unit vectors) of the geodesic emanating from q toward the interior of Q .²*

Then for any $u \in \Omega$ there exists a b -projecting shortest path which forms an angle $\phi \leq \pi/2$ with the vector $i_(u)$.*

The proof of Lemma 2 will be given in §3. Meanwhile we continue with the proof of the theorem.

Obviously Ω is homeomorphic to an open $(m - 1)$ -dimensional circle when $q \in \partial Q$. Denote by C the cut locus of Q from the point q . Let $u \in \Omega$ and $z(u) > 0$ be the minimum number such that $\exp_q(z(u) \cdot u) \in C \cup \partial Q$. $z(u)$ exists since the set $C \cup \partial Q$ is closed (because $Q \setminus (C \cup \partial Q)$ is obviously open).

We want to show first that

$$(2.3) \quad z(u) \leq \frac{1}{2} L_\chi, \quad u \in \Omega.$$

By Lemma 2, there exists a b -projecting shortest path ba which forms an angle $\phi \leq \pi/2$ with the curve $g: [0, z(u)] \rightarrow T$ given by the formula

$$(2.4) \quad g(l) = i(\exp_q(lu)).$$

Applying the inequality (1.4) in [3] (where $\bar{L}_0(\pi/2)$ is obviously $\frac{1}{2} L_\chi$) to the curve g of curvature $\leq \chi$ and the mentioned path ba we obtain (2.3).

Let us put $\Phi = \cup lu$, where $u \in \Omega$ and $0 < l < z(u)$. It is easy to see that

²We mean that all points of the geodesic sufficiently close to q and different from q lie in $\text{int } T$. If $q \notin \partial Q$, then $\Omega = S^m$.

for any $p \in \text{int } Q$, $p \neq q$, the minimum geodesic qp emanates from q toward the interior of Q . Therefore

$$\Psi \stackrel{\text{def}}{=} \exp_q(\Phi) = Q \setminus (q \cup \partial Q \cup C)$$

and the mapping $\exp_q$ restricted to Φ is one-to-one. By an argument contained in Lemma 8 in [2] we can see that the set C has m -dimensional measure zero. Therefore

$$(2.5) \quad V(Q) = V(\Psi),$$

where $V(\cdot)$ means volume.

Let $\tilde{q}$ be the center of the ball B in the space $\tilde{Q} \stackrel{\text{def}}{=} P^m(k_s - (n - m)\chi^2)$ and $j: Q_q \rightarrow \tilde{Q}_{\tilde{q}}$ be an isometry. Let us put

$$(2.6) \quad f = \exp_{\tilde{q}} \circ j \circ \exp_q^{-1}: \Psi \rightarrow \tilde{Q}.$$

By (2.3), the set $\tilde{\Psi} \stackrel{\text{def}}{=} f(\Psi) \subset B$ so that

$$(2.7) \quad V(\tilde{\Psi}) \leq V(B).$$

On the strength of (2.5) and (2.7), the proof will be completed if we show that

$$(2.8) \quad V(\Psi) \leq V(\tilde{\Psi}).$$

To show this it is enough to establish that the induced mapping $f_*: T\Psi \rightarrow T\tilde{\Psi}$ does not decrease the length of tangent vectors. Really, if it is proved then the mapping f does not decrease volumes since, for any point $p \in \Psi$, the mapping $f_{*p}: \Psi_p \rightarrow \tilde{\Psi}_{\tilde{p}}$ ($\tilde{p} \stackrel{\text{def}}{=} f(p)$), like any linear transformation, can be reduced to expansions into m pair-wise orthogonal directions and to orthogonal transformations.

Let $p \in \Psi$ and a vector $v \in \Psi_p$. We need to show that $|f_*(v)| \geq |v|$. Let the vectors v^- and $v^\perp$ be such that $v = v^- + v^\perp$, $v^- \perp v^\perp$, and v^- is directed along the shortest path pq . It follows easily from the definition of f that $f_*(v^-)$ is directed along $\tilde{p}\tilde{q}$, $|f_*(v^-)| = |v^-|$, and $f_*(v^\perp) \perp f_*(v^-)$. Since $f_*(v) = f_*(v^-) + f_*(v^\perp)$, it will be enough to prove that $|f_*(v^\perp)| \geq |v^\perp|$.

Let $p(t)$, $t \in [0, \varepsilon]$, be a curve such that $p(0) = p$, $\dot{p}(0) = v^\perp$ and $qp(t) = qp$. We put $\tilde{p}(t) = f(p(t))$. Obviously, $\tilde{q}\tilde{p}(t) = qp(t) = qp$, and $\angle \tilde{p}\tilde{q}\tilde{p}(t) = \angle pqp(t)$. By Rauch comparison theorem we have: $|\dot{\tilde{p}}(0)| \geq |\dot{p}(0)|$, i.e. $|f_*(v^\perp)| \geq |v^\perp|$. This completes the proof of the theorem.

3. Proof of Lemma 2. Suppose otherwise. That is suppose that some $u \in \Omega$ forms an angle $> \pi/2$ with any b -projecting shortest path. We consider the curve g given by (2.4). Let $l_i \rightarrow_\infty 0$, $l_i > 0$, and $b_i = g(l_i)$, $i = 1, 2, \dots$. Denote by $b_i a_i$ a b_i -projecting shortest path. By a simple compactness argument (selection of a subsequence), we may regard the sequence $b_i a_i$ as converging to a b -projecting shortest path ba .

We first want to prove the following statement

(α) there are no conjugate points on the segment ba .

Let us extend ba beyond its end a (as a geodesic) to a close point d such

that the extension ad is still a shortest path and $S \cap T = a$ where S is the sphere of radius ad with the center at d (see the figure). If (α) is not true, then there exists an arc $\widetilde{bd}$ satisfying

$$(3.1) \quad l(\widetilde{bd}) < ba + ad$$

where $l(\cdot)$ means the length. In what follows we will denote by $\widetilde{\cdot}$ parts of the arc $\widetilde{bd}$. Let $e \in \widetilde{bd}, \widetilde{be} \subset T, e \in \Gamma$ and let $h \in \widetilde{bd} \cap S$. We have:

$$(3.2) \quad l(\overline{be}) \leq l(\overline{bh}),$$

$$(3.3) \quad l(\overline{hd}) \geq ad.$$

Subtracting (3.3) from (3.1) we obtain that $l(\widetilde{bh}) < ba$. Combining it with (3.2) we find that $l(\widetilde{be}) < ba$ which is impossible by definition of b -projecting shortest path ba . So, (α) has been proved.

For sufficiently large i the inequality $bb_i < ba \leq ba_i$ holds. Therefore on $b_i a_i$ there exists a point c_i such that $bc_i = ba$. Let us connect b and c_i with a shortest path bc_i and consider the triangle $bb_i c_i$ (see the figure). The direction $u_i \in \Omega$ of its side bb_i converges to u . The relation $a_i \rightarrow a$ obviously implies $c_i \rightarrow a$. This fact together with (α) means that the direction v_i of the side bc_i converges to the direction v of ba .

Let us regard the directions u_i , v_i and the length bb_i as three independent variables. As soon as $bc_i = ba$ does not depend on i and because of (α), there exists $\epsilon > 0$ such that the length $b_i c_i$ is a regular function $\lambda(u_i, v_i, bb_i)$ in the compact region σ : $|u_i - u| \leq \epsilon$, $|v_i - v| \leq \epsilon$, $|bb_i| \leq \epsilon$. (Negative bb_i means the displacement in the direction $-u_i$.) By Hadamard Lemma,

$$\begin{aligned} \lambda(u_i, v_i, bb_i) &= \lambda(u_i, v_i, 0) + \frac{\partial \lambda}{\partial bb_i}(u_i, v_i, 0) \cdot bb_i \\ &\quad + \theta(u_i, v_i, bb_i) \cdot (bb_i)^2 \end{aligned} \quad (3.4)$$

where the function θ is regular in σ and therefore bounded. Let C be such that $|\theta| < C$. For sufficiently small ε , the segment bc_i being close to ba does

not contain conjugate points. Therefore $\partial\lambda(u_i, v_i, 0)/\partial bb_i = -\langle u_i, v_i \rangle$. Obviously, $\lambda(u_i, v_i, 0) = ba$.

Now, for large i , the point $(u_i, v_i, bb_i) \in \sigma$ and we get from (3.4) that

$$(3.5) \quad b_i c_i > ba + bb_i(-\langle u_i, v_i \rangle - C \cdot bb_i).$$

By our original contrary assumption, $\langle u, v \rangle < 0$. Therefore $\lim_{i \rightarrow \infty} \langle u_i, v_i \rangle = \langle u, v \rangle < 0$. Since $bb_i \rightarrow 0$, we find from (3.5) that $b_i c_i > ba$ for sufficiently large i . Consequently, $b_i a_i > ba$. But then the point q ($q \in i^{-1}(b)$) is not remote. This contradiction proves the Lemma.

BIBLIOGRAPHY

1. B. V. Dekster, *Estimates of the volume of a region in a Riemannian space*, Math. USSR-Sb. **17** (1972), 61–87.
2. ———, *An inequality of the isoperimetric type for a domain in a Riemannian space*, Math. USSR-Sb. **19** (1973), 257–274.
3. ———, *Estimates of the length of a curve*, J. Differential Geometry **12** (1977), 99–115.
4. D. Gromoll, W. Klingenberg and W. Meyer, *Riemannische Geometrie im Grossen*, Lecture Notes in Math., vol. 55 Springer-Verlag, Berlin and New York, 1968.

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