MULTIPLICATION RULES FOR POLYNOMIALS

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ABSTRACT. It is proved that the polynomial solutions of the functional equation

$$F(z)F(z + 1/\sqrt{a}) = F(\sqrt{a}z^2 + (b/\sqrt{a} + 1)z + c/\sqrt{a})$$

are precisely $(az^2 + bz + c)^n$ if $b^2 - 4ac \neq 0$ and $(\sqrt{a}z + b/2\sqrt{a})^n$ if $b^2 - 4ac = 0$.

Let $F(z) = az^2 + bz + c$, where $a, b, c \in \mathbb{C}$ and $a \neq 0$. Then $F(z)$ satisfies the "multiplication rule"

$$(1) \quad F(z)F(z + 1/\sqrt{a}) = F(\sqrt{a}z^2 + (b/\sqrt{a} + 1)z + c/\sqrt{a}).$$

In particular, if $a = 1$, then $F(z)F(z + 1) = F(z^2 + (b + 1)z + c)$. From this observation, Cohen, Erdős, and Nathanson [2] deduced that if $F(z) = z^2 + bz + c$ with $b, c \in \mathbb{Z}$, and if $F(n) \geq 1$ for all $n \geq 1$, then infinitely many terms of the sequence $\{F(n)\}_{n=1}^{\infty}$ are "composite" in the sense that they factor into the product of strictly smaller terms of the sequence.

If $F(z) = z^2 + 1$, then the multiplication rule (1) implies that $F(z)F(z + 1) = F(z^2 + z + 1)$. H. N. Shapiro [3] proved the converse: If $F(z)$ is a nonzero polynomial such that $F(z)F(z + 1) = F(z^2 + z + 1)$, then $F(z) = (z^2 + 1)^n$. In this note I extend Shapiro's result to the general multiplication rule (1): If $b^2 - 4ac \neq 0$, then the only nonzero polynomial solutions of (1) are $(az^2 + bz + c)^n$; if $b^2 - 4ac = 0$, then the only nonzero polynomial solutions of (1) are $(\sqrt{a}z + b/2\sqrt{a})^n$. This result depends on the following Lemma, first proved by the author for real d , and then extended by B. J. Birch [1] to complex d with a remarkably simple proof.

LEMMA. Let $d \in \mathbb{C}$, and let S_d be a finite, nonempty set of complex numbers such that $z \in S_d$ implies $z^2 \pm z - d \in S_d$. Then $\{\pm\sqrt{d}\} \subseteq S_d$. Moreover, $S_0 = \{0\}$.

PROOF. The set S_d is finite and nonempty, and so there exists $z \in S_d$ with $|z|$ maximal. Since $z^2 \pm z - d \in S_d$, it follows that $|z| \geq \max\{|z^2 + z - d|, |z^2 - z - d|\}$. But $(z^2 + z - d) - (z^2 - z - d) = 2z$, hence $|z| = |z^2 + z - d| = |z^2 - z - d|$. This is possible only if $z^2 - d = 0$, hence $z = \sqrt{d}$ or $z = -\sqrt{d}$. But if $\sqrt{d}$ (resp. $-\sqrt{d}$) belongs to S_d , then so does $-\sqrt{d}$ (resp. $\sqrt{d}$).

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$\sqrt{d}$). Moreover, if $z \in S_0$, then $|z| \leq 0$, and so $S_0 = \{0\}$.

THEOREM 1. *Let $a, b, c \in \mathbb{C}$, $a \neq 0$, and let $F(z)$ be a nonzero polynomial such that*

$$F(z)F(z + 1/\sqrt{a}) = F(\sqrt{a}z^2 + (b/\sqrt{a} + 1)z + c/\sqrt{a}).$$

Then for some $n \geq 0$,

$$F(z) = \begin{cases} (az^2 + bz + c)^n & \text{if } b^2 - 4ac \neq 0, \\ (\sqrt{a}z + b/2\sqrt{a})^n & \text{if } b^2 - 4ac = 0. \end{cases}$$

PROOF. If $F(z) = k$ is a nonzero constant satisfying (1), then $k^2 = k$ and so $F(z) = 1$. If $F(z)$ is a nonconstant polynomial satisfying (1), then $G(z) = F(z/\sqrt{a} - b/2a)$ satisfies

$$(2) \quad G(z)G(z + 1) = G(z^2 + z - d),$$

where $d = (b^2 - 4ac)/4a$. The set S_d of roots of the polynomial $G(z)$ is a finite nonempty set of complex numbers, and (2) implies that S_d is closed under $z \mapsto z^2 \pm z - d$. By the Lemma, S_d contains $\pm\sqrt{d}$. If $d \neq 0$, then $G(z)$ can be written in the form $G(z) = (z^2 - d)^n H(z)$, where $n \geq 1$ and not both $\pm\sqrt{d}$ are roots of $H(z)$. Since both $G(z)$ and $(z^2 - d)^n$ satisfy the functional equation (2), it follows that $H(z)$ also satisfies (2). But if $H(z)$ is nonconstant, then the set of roots of $H(z)$ must also contain both $\pm\sqrt{d}$ which is impossible. Therefore, $H(z)$ is constant, hence $H(z) = 1$, and $G(z) = (z^2 - d)^n$. Then

$$F(z) = G(\sqrt{a}z + b/2\sqrt{a}) = (az^2 + bz + c)^n.$$

If $d = 0$, then $S_0 = \{0\}$ and $G(z) = z^n$, hence

$$F(z) = G(\sqrt{a}z + b/2\sqrt{a}) = (\sqrt{a}z + b/2\sqrt{a})^n.$$

This proves the theorem.

If $F(z) = az^2 + bz + c$ with $a \neq 0$, then the multiplication rule (1) implies that

$$(3) \quad F(z)F(z + 1/\sqrt{a}) = F(z + F(z)/\sqrt{a}).$$

This observation has the following converse.

THEOREM 2. *Let $a \neq 0$ and let $F(z)$ be a nonconstant polynomial such that (3) holds. Then $F(z) = az^2 + bz + c$.*

PROOF. Let $F(z)$ satisfy (3). If $F(z)$ has degree n , then the left side of (3) has degree $2n$ and the right side of (3) has degree n^2 . Therefore, $n = 2$. If $F(z)$ has leading coefficient A , then the left side of (3) has leading coefficient A^2 and the right side of (3) has leading coefficient $A(A/\sqrt{a})^2 = A^3/a$. Therefore, $A = a$, and so $F(z) = az^2 + bz + c$ for some $b, c \in \mathbb{C}$. This proves the Theorem.

PROBLEMS. The function $F(z)$ satisfies a multiplication rule if there exist

polynomials $\alpha(z), \beta_1(z), \dots, \beta_t(z)$ with $t \geq 2$ and degree $\alpha \geq 2$ such that $F(\alpha(z)) = F(\beta_1(z)) \cdots F(\beta_t(z))$. Which polynomials $F(z)$ satisfy a multiplication rule? In particular, does $z^n + 1$ satisfy a multiplication rule for any $n \geq 3$? Similar questions can be asked about more general multiplication rules of the form

$$F(\alpha_1(z))F(\alpha_2(z)) \cdots F(\alpha_s(z)) = F(\beta_1(z)) \cdots F(\beta_t(z))$$

where $\alpha_i(z) \neq \beta_j(z)$ for $1 \leq i \leq s$ and $1 \leq j \leq t$, and degree $\alpha_1 + \cdots + \alpha_s \geq 2$.

The only polynomial solutions of the functional equation $F(z)F(z+1) = F(z+F(z))$ are the monic quadratic polynomials. What are the nonpolynomial solutions of this equation?

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