ONE-RELATOR GROUPS HAVING A FINITELY PRESENTED NORMAL SUBGROUP¹

A. KARRASS AND D. SOLITAR

ABSTRACT. A classification is given for one-relator groups having a finitely presented normal subgroup of infinite index.

The main result is the following:

THEOREM. *Let G be a one-relator group having a (nontrivial) finitely presented normal subgroup H of infinite index. Then G is torsion free, has two generators, and is an infinite cyclic or infinite dihedral extension of a finitely generated free group N ; moreover, N can be chosen so that $H < N$ if H is not cyclic, and $H \cap N = (1)$ if H is cyclic.*

PROOF. The group G has a torsion free subgroup P of finite index (see, for example [4]), and the cohomology dimension of $P \leq 2$ (by Bieri [2, Theorem 7.7]). Clearly, $H \cap P$ is a finitely presented normal subgroup ($\neq 1$) of infinite index in P . Therefore (by Bieri [2, Corollaries 8.6 and 8.7]) $H \cap P$ is free and $P/H \cap P$ is a finite extension of degree j , of a free group of rank $r < \infty$.

Let $\chi(G)$ be the (rational) Euler characteristic of G . (For the proofs of the properties of χ that we use, see Chiswell [3] including the remark added in proof.) Then

$$\chi(P) = \chi(H \cap P) \cdot \chi(P/H \cap P). \quad (1)$$

Now if r_1, r_2 are the numbers of generators of $G, H \cap P$ respectively, and n is the maximum finite order of the elements of G , then $\chi(G) = 1 - r_1 + 1/n$, $\chi(H \cap P) = 1 - r_2$, and $\chi(P/H \cap P) = (1 - r)/j$. Clearly, $\chi(G) \leq 0$, $\chi(H \cap P) \leq 0$, and $\chi(P/H \cap P) \leq 0$. Since $\chi(P) = [G : P] \cdot \chi(G) \leq 0$, and $\chi(H \cap P) = [H : H \cap P] \cdot \chi(H) \leq 0$, it follows from equation (1) that $\chi(G) = 0$, and either $\chi(H) = 0$ or $\chi(P/H \cap P) = 0$. Hence, $r_1 = 2, n = 1$, i.e., G is torsion free and has two generators, and so we may assume $G = P$. Moreover, either $r_2 = 1$, or $r = 1$.

Suppose $r = 1$. Then G/H is a finite extension of an infinite cyclic group, and so $G/H = (A_1 * B_1; U_1)$ where U_1 is finite and $A_1: U_1 = B_1: U_1 = 2$, or $G/H = \langle t_1, K_1; t_1 K_1 t_1^{-1} = K_1 \rangle$ where K_1 is finite (by Stallings [8]). Hence

Received by the editors June 17, 1977.

AMS (MOS) subject classifications (1970). Primary 20F05, 20E30.

Key words and phrases. One-relator groups.

¹This research was supported in part by grants from the National Research Council of Canada, Grant No. A5602 and Grant No. A5614.

© American Mathematical Society 1978

$G = (A * B; U)$ where H is of finite index in U , which is therefore free, and $A : U = B : U = 2$; or $G = \langle t, K; tKt^{-1} = K \rangle$ where H is of finite index in K , which is therefore free. Hence, G is an infinite cyclic or infinite dihedral extension of a finitely generated free group which contains H .

Now suppose $r_2 = 1$, i.e., H is a normal cyclic subgroup of G . By Bieri [2, Corollary 8.7], G is a treed HNN group with finitely many vertices and edges each of which is infinite cyclic and contains H . We first map the tree product base of G into an infinite cyclic group as follows: Map the generator a of a fixed vertex into 1 in the additive group of rationals and extend the mapping to its neighboring vertices in such a way that the amalgamation relations are satisfied; e.g., if $a^p = b^q$ then b is mapped into p/q . Continuing in this way with the neighbors of b , etc., we obtain a homomorphism of the base which is 1-1 on each vertex into a finitely generated and hence cyclic subgroup $\text{gp}(c)$ of the rationals.

We now extend this map to a mapping of G into the infinite dihedral group $D = \langle m, c; m^2, mcm^{-1} = c^{-1} \rangle$. If t is a stable letter of G and $H = \text{gp}(h)$, then $tht^{-1} = h^\epsilon$, $\epsilon = \pm 1$; if $\epsilon = 1$, map t into c and if $\epsilon = -1$, map t into m . If the relation corresponding to t is $ta^\alpha t^{-1} = b^\beta$, then the corresponding edge groups lying in the vertex groups are generated by a^α and b^β . Since h lies in each edge group, there exist u and v not zero with $h = a^{au} = b^{\beta v}$, and so $h^\epsilon = tht^{-1} = ta^{au}t^{-1} = b^{\beta u} = b^{\epsilon\beta v}$ and $u = \epsilon v$. Also if h goes into the rational number δc then $a^\alpha \rightarrow (\delta c)/u$ and $b^\beta \rightarrow (\delta c)/v$. Hence $ta^\alpha t^{-1}$ and b^β are each mapped into $\delta c/\epsilon u$.

We thus have a homomorphism of G into D which is 1-1 on each vertex. The following lemma applies and we have that G is an infinite cyclic or infinite dihedral extension of the kernel N where N is a finitely generated free group. $\square$

LEMMA. *Let G be a treed HNN group with finitely many edges and let α be a homomorphism of G which is 1-1 on each vertex of G and is such that the image of each edge is of finite index in $\alpha(G)$. Then the kernel N of α is a finitely generated free group.*

PROOF OF LEMMA. We use induction on the number of edges of G . The result is trivially true if there are no edges. If there is a stable letter t , then $G = \langle t, K; tLt^{-1} = M \rangle$ where K is a treed HNN group having one fewer edge than G . Since α is 1-1 on the vertices of G , $N \cap L = 1$. Hence by the subgroup theorem in [5], N is the free product of a free group F and factors $EKE^{-1} \cap N$ where E ranges over a double coset representative system for $G \bmod(N, K)$. By inductive hypothesis, $EKE^{-1} \cap N = E(K \cap N)E^{-1}$ is a finitely generated free group. Hence N is a free group. Moreover, since $\alpha(G) : \alpha(L) = G : NL < \infty$, it follows that $[G : (N, L)], [G : (N, K)] < \infty$. Therefore E ranges over a finite set, and the rank of the free part F of N which is $[G : (N, L)] - [G : (N, K)] + 1$ is finite. Thus N is a finitely generated free group.

If there is no stable letter, then $G = (A * B; U)$ where A is an extremal

vertex of G and B is a treed HNN group having one fewer edge than G . A similar argument using the subgroup theorem for amalgamated products in [6] applies in this case to show again that N is a finitely generated free group. $\square$

COROLLARY 1. *Let G, H be as in the theorem. Then G is residually finite and G'' is free.*

PROOF. G is a finite extension of an infinite cyclic extension of a finitely generated free group and so G is residually finite (see, e.g., Baumslag [1]). Also since Z and $Z_2 * Z_2$ are metabelian, G'' is free. $\square$

COROLLARY 2. *Let G be a group having an infinite cyclic normal subgroup. Then the following are equivalent:*

- (i) G is finitely presented with $cdG \leq 2$;
- (ii) G is a treed HNN group with infinite cyclic vertices and finitely many edges.
- (iii) G is a torsion free infinite cyclic or infinite dihedral extension of a finitely generated free group.

PROOF. That (i) implies (ii) and (iii), has been shown in the proof of the theorem. If (iii) holds, then G has the form $\langle t, K; tKt^{-1} = K \rangle$ or $(A * B; U)$ where K, A, B are free. Hence $cdG \leq 2$ (see, e.g., Bieri [2, Propositions 6.1 and 6.12]), and (i) holds. Similarly (ii) implies (i). $\square$

Note that (ii) above implies that every finitely generated subgroup of G is finitely presented (see, for example [6] and [7]), and hence every finitely generated normal subgroup of infinite index in G is free.

Note also that the argument in the proof of the theorem applies to any group G of $cd \leq 2$ with $\chi(G) \leq 0$, such as a finite extension or a subgroup of finite index of a one-relator group, and yields the same conclusions except that G may have more than two generators.

Finally, we consider two examples suggested by A. Pietrowski. First let

$$G_1 = (\langle a, b \rangle * \langle c, d \rangle; b = c, a^2 = dcd^{-1}, aba^{-1} = d^2),$$

which is the free product of two free groups with subgroups of index two amalgamated and therefore is an infinite dihedral extension of its amalgamated subgroup. Eliminating b and c , we obtain

$$G_1 = \langle a, d; a^{-1}d^2a = d^{-1}a^2d \rangle.$$

Replacing d by ax results in

$$G_1 = \langle a, x; (xa)^2x^{-1}a^{-2}x \rangle,$$

or equivalently

$$G_1 = \langle a, x_0, x_1, x_2; x_0x_1x_2^{-1}x_0, ax_0a^{-1} = x_1, ax_1a^{-1} = x_2 \rangle.$$

Thus G_1 is a one-relator group which is an infinite dihedral extension of a finitely generated free group. However, G_1 is not an infinite cyclic extension of a free group; for, since the relator $(xa)^2x^{-1}a^{-2}x$ is not a commutator, there

is a unique normal subgroup P such that G_1/P is infinite cyclic, viz., $P = \text{gp}(x)^{G_1}$, which is a proper ascending union of free groups of rank two, so that P is not free.

On the other hand,

$$G_2 = (\langle a, b \rangle * \langle c, d \rangle; b = c^2, a^2 = cd, aba^{-1} = dc^{-1}),$$

is the free product of two free groups with subgroups of index two amalgamated, and therefore an infinite dihedral extension of its amalgamated subgroup. Now G_2 is the one-relator group $G_2 = \langle a, c; ac^2a^{-1}ca^{-2}c \rangle$. Letting $a = c^2x$, we obtain a relator with c -exponent sum zero, and we see that $\text{gp}(x)^{G_2}$ is a finitely generated free group, and so G_2 is both an infinite cyclic and an infinite dihedral extension of a finitely generated free group.

Results similar to the main theorem have been obtained independently by R. Bieri.

REFERENCES

1. G. Baumslag, *Finitely generated cyclic extensions of free groups are residually finite*, Bull. Austral. Math. Soc. **5** (1971), 87–94.
2. R. Bieri, *Homological dimension of discrete groups*, Queen Mary College Mathematics Notes, London, 1976.
3. I. M. Chiswell, *Euler characteristics of groups*, Math. Z. **147** (1976), 1–11.
4. J. Fischer, A. Karrass and D. Solitar, *On one-relator groups having elements of finite order*, Proc. Amer. Math. Soc. **33** (1972), 297–301.
5. A. Karrass, A. Pietrowski and D. Solitar, *An improved subgroup for HNN groups with some applications*, Canad. J. Math. **26** (1974), 214–224.
6. A. Karrass and D. Solitar, *Subgroups of HNN groups and groups with one defining relation*, Canad. J. Math. **23** (1971), 627–643.
7. ———, *The subgroups of a free product of two groups with an amalgamated subgroup*, Trans. Amer. Math. Soc. **149** (1970), 227–255.
8. J. Stallings, *Groups of cohomological dimension one*, Proc. Sympos. Pure Math., vol. 17, Amer. Math. Soc., Providence, R. I., 1970, pp. 124–128.

DEPARTMENT OF MATHEMATICS, YORK UNIVERSITY, DOWNSVIEW, ONTARIO, CANADA