AN ALGEBRA NORM ON l^2 WHOSE RESTRICTION TO l^1 IS THE USUAL l^1 -NORM

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ABSTRACT. Let p_1 and p_2 be two real numbers such that $p_2 > p_1 > 1$. We prove that there exists an algebra norm on l^{p_2} which extends the l^{p_1} -norm if and only if $p_2 < 2p_1$.

1. Introduction. Let $\mathcal{Q}$ be a (nonunital) Banach algebra and let $\mathcal{Q}^2$ be the linear space spanned by the set of all products of elements of $\mathcal{Q}$. If $\mathcal{Q}^2$ is not closed in $\mathcal{Q}$, or if $\mathcal{Q}^2$ is infinite codimensional in $\mathcal{Q}$, there is an easy, well-known way to construct discontinuous homomorphisms from $\mathcal{Q}$: the above conditions imply the existence of a discontinuous linear form L in $\mathcal{Q}$ whose kernel contains $\mathcal{Q}^2$. If b is a nonzero element of the one-dimensional nilpotent algebra $\mathcal{B}$, the map $\Psi_L: x \rightarrow L(x)b$ is clearly a discontinuous homomorphism from $\mathcal{Q}$ into $\mathcal{B}$. (See [1, §3, Example 1]; completeness plays, of course, no role in this construction.) This trivial example shows that when studying automatic continuity properties for a homomorphism Ψ from such a nonunital Banach algebra $\mathcal{Q}$ into an arbitrary Banach algebra, it is only possible to expect results concerning $\Psi|_{\mathcal{Q}^2}$ (a general account concerning automatic continuity can be found in [5] and [15]).

Now let $\mathcal{Q}$ be a (nonunital) normed algebra, not necessarily complete; $\mathcal{Q}^2$ being defined as above, put, for every $u \in \mathcal{Q}^2$,

$$\|u\|_{\otimes} = \inf \sum_{i=1}^k \|x_i\| \cdot \|y_i\|,$$

the infimum being taken over all finite families $\{(x_1, y_1) \cdots (x_k, y_k)\}$ of elements of $\mathcal{Q} \times \mathcal{Q}$ satisfying $u = \sum_{i=1}^k x_i y_i$. We prove here that there exists an algebra norm λ on $\mathcal{Q}$ satisfying, for every $u \in \mathcal{Q}^2$, $\lambda(u) = \|u\|_{\otimes}$ (this fact was mentioned without proof in [7]). Though elementary, this result may have some "heuristic" interest for automatic continuity. Indeed when $\sup_{u \in \mathcal{Q}^2 \setminus \{0\}} \|u\|_{\otimes} / \|u\| = +\infty$, it shows that the only general automatic continuity results which may be expected for a homomorphism Ψ from $\mathcal{Q}$ into an arbitrary Banach algebra must concern, in fact, the restriction of Ψ to $\mathcal{Q}^2$ when $\mathcal{Q}^2$ is equipped with the topology defined by the norm $\|\cdot\|_{\otimes}$ instead of the topology induced by the given topology of $\mathcal{Q}$. In other terms, automatic

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continuity results for a homomorphism Ψ from such an algebra concern the continuity of the bilinear map $(x, y) \rightarrow \Psi(xy)$ and not the continuity of the homomorphism Ψ itself.

2. An extension theorem for some algebra norms defined on $\mathcal{Q}^2$.

THEOREM 2-1. *Let $\mathcal{Q}$ be a normed algebra and let μ be an algebra norm defined on $\mathcal{Q}^2$. If there exist two positive real numbers k_1 and k_2 satisfying $\|u\| \leq k_1 \mu(u)$ ($u \in \mathcal{Q}^2$), $\mu(xy) \leq k_2 \|x\| \cdot \|y\|$ ($x \in \mathcal{Q}, y \in \mathcal{Q}$), then there exists an algebra norm λ on $\mathcal{Q}$ such that $\lambda|_{\mathcal{Q}^2} = \mu$.*

PROOF. We can suppose k_1 and k_2 greater than 1. Let T be a linear map from $\mathcal{Q}$ onto $\mathcal{Q}^2$ such that $T \circ T = T$ (the existence of such a map is a well-known consequence of Zorn's lemma). For $x \in \mathcal{Q}$, put $\lambda(x) = \mu[T(x)] + k_1 k_2 \|x - T(x)\|$. We clearly obtain a linear norm on $\mathcal{Q}$.

For $u \in \mathcal{Q}^2$, there exists $x \in \mathcal{Q}$ such that $u = T(x)$; so $u = (T \circ T)(x) = T(u)$ and $\lambda(u) = \mu(u)$, hence $\lambda|_{\mathcal{Q}^2} = \mu$.

Also, for $y \in \mathcal{Q}, z \in \mathcal{Q}$ we have, as μ is an algebra norm on $\mathcal{Q}^2$:

$$\mu[T(y) \cdot T(z)] \leq \mu[T(y)] \mu[T(z)]$$

and

$$\begin{aligned} \lambda(yz) &= \mu(yz) \leq \mu[T(y) \cdot T(z)] + \mu[(y - T(y)) \cdot T(z)] \\ &\quad + \mu[T(y)(z - T(z))] + \mu[(y - T(y))(z - T(z))] \\ &\leq \mu[T(y)] \cdot \mu[T(z)] + k_2 \|y - T(y)\| \cdot \|T(z)\| \\ &\quad + k_2 \|T(y)\| \cdot \|z - T(z)\| + k_2 \|y - T(y)\| \|z - T(z)\| \\ &\leq \mu[T(y)] \mu[T(z)] + k_1 k_2 \|y - T(y)\| \mu[T(z)] + k_1 k_2 \mu[T(y)] \\ &\quad \cdot \|z - T(z)\| + k_1^2 k_2^2 \|y - T(y)\| \cdot \|z - T(z)\|. \end{aligned}$$

Finally we obtain

$$\lambda(yz) \leq \lambda(y) \lambda(z) \quad (y, z \in \mathcal{Q}^2).$$

Hence λ is an algebra norm on $\mathcal{Q}^2$ and the theorem is proved.

It follows immediately from the definition of the norm $\|\cdot\|_{\otimes}$ that we have

$$\|u\| \leq \|u\|_{\otimes} \quad (u \in \mathcal{Q}^2), \quad \|xy\|_{\otimes} \leq \|x\| \cdot \|y\| \quad (x, y \in \mathcal{Q})$$

and these inequalities imply

$$\|uv\|_{\otimes} \leq \|u\|_{\otimes} \cdot \|v\|_{\otimes} \quad (u, v \in \mathcal{Q}^2).$$

Thus we have the following

COROLLARY 2-2. *For every normed algebra $\mathcal{Q}$, there exists another algebra norm λ on $\mathcal{Q}$ satisfying $\lambda(u) = \|u\|_{\otimes}$ ($u \in \mathcal{Q}^2$).*

3. An application to extension problems for l^p -norms. Let H be a Hilbert space and let $p > 0$. Recall that $\mathcal{L}^p(H)$ is the algebra of elements u of $\mathcal{L}(H)$ such that $\sum_{n=1}^{\infty} |\alpha_n(u)|^p < +\infty$, where $\alpha_n(u)$ is the sequence of eigenvalues

(listed according to their multiplicities) of the positive hermitian operator $(uu^*)^{1/2}$. For $u \in \mathcal{L}^p(H)$ denote as usual

$$\|u\|_p = \left[\sum_{n=1}^{\infty} |\alpha_n(u)|^p \right]^{1/p}.$$

For $p \geq 1$ this formula defines an algebra norm on $\mathcal{L}^p(H)$ for which $\mathcal{L}^p(H)$ is complete. We have

$$\begin{aligned} [\mathcal{L}^p(H)]^2 &= \mathcal{L}^{p/2}(H) \quad (0 < p \leq +\infty); \\ \|u\|_{p \otimes} &= \|u\|_1 \quad (u \in \mathcal{L}^{p/2}(H)) \quad \text{if } 1 \leq p \leq 2 \end{aligned}$$

and

$$\|u\|_{p \otimes} = \|u\|_{p/2} \quad (u \in \mathcal{L}^{p/2}(H)) \quad \text{if } p \geq 2$$

(these well-known facts are easy consequences of the "Hölder inequality for operators" [12]).

Now take $p \geq 1$. It is stated in [7] that for every homomorphism from $\mathcal{L}^p(H)$ into a Banach algebra, the bilinear map $(x, y) \rightarrow \Psi(x, y)$ is continuous (a proof of this result is outlined in [7] and I shall not give further details here because the method used in [7] is only a slight improvement of B. E. Johnson's ideas [13]). This means that $\Psi|_{\mathcal{L}^{p/2}(H)}$ is continuous when $\mathcal{L}^{p/2}(H)$ is equipped with the norm $\|\cdot\|_1$ if $1 \leq p \leq 2$ or with the norm $\|\cdot\|_{p/2}$ if $p \geq 2$, and Corollary 2-2 shows then that this result is best possible.

The situation is much more complicated if we consider the commutative Banach algebra l^p (the multiplication being coordinatewise and the norm being the usual l^p -norm $\|\cdot\|_p$): G. Dales and the author proved recently that assuming the continuum hypothesis there exists a discontinuous homomorphism from $\mathcal{C}(K)$, the algebra of continuous valued functions over an arbitrary infinite compact space K . (See [3], [8], [9], [10]; a short summary of both constructions can be found in [6].) It is possible to prove more generally by similar methods that every nonmaximal modular prime ideal of a commutative Banach algebra $\mathcal{Q}$ is the kernel of a discontinuous homomorphism from $\mathcal{Q}$. (See [4] and [11].) So there may exist a homomorphism Ψ from l^p into a Banach algebra such that the bilinear map $(x, y) \rightarrow \Psi(x, y)$ is discontinuous.

Nevertheless we have the following:

THEOREM 3-1. *Let $p_1 \geq 1, p_2 > p_1$. There exists an algebra norm on l^{p_2} whose restriction to l^{p_1} is the l^{p_1} -norm $\|\cdot\|_{p_1}$ if and only if $p_2 \leq 2p_1$.*

Moreover, the quotient algebra l^{p_2}/l^{p_1} is not normable with an algebra norm if $p_2 > 2p_1$.

PROOF. For $n \in N$ let e_n be the sequence whose n th term is equal to one, all the others being zero. Put $f_n = \sum_{i=1}^n e_i$; we have $f_n \in l^p$, $\|f_n\|_p = n^{1/p}$ ($n \in N, p \geq 1$). Note that f_n is idempotent.

By a classical automatic continuity result [1, Theorem 2-3], we can then

associate to every algebra norm λ defined over l^{p_2} a positive real K_λ satisfying

$$\lambda(f_n) \leq K_\lambda (\|f_n\|_{p_2})^2 = K_\lambda \cdot n^{2/p_2} \quad (n \in N).$$

If, furthermore, $p_2 > 2p_1$, we have

$$\liminf_{n \rightarrow \infty} \frac{\|f_n\|_{p_1}}{(\|f_n\|_{p_2})^2} = \liminf_{n \rightarrow \infty} n^{1/p_1 - 2/p_2} = +\infty,$$

and in this case there does not exist any algebra norm on l^{p_2} whose restriction to l^{p_1} is the l^{p_1} -norm (more generally there does not exist in this case any algebra norm over l^{p_2} whose restriction to l^{p_1} defines the same topology as the l^{p_1} -norm). On the other hand, a straightforward verification based on Hölder's inequality shows that we have

$$(l^{2p_1})^2 = l^{p_1}, \quad \|u\|_{2p_1 \otimes} = \|u\|_{p_1} \quad (u \in l^{p_1}).$$

By Corollary 2-2 there exists an algebra norm λ over l^{2p_1} such that $\lambda(u) = \|u\|_{p_1}$ ($u \in l^{p_1}$). As l^{2p_1} contains l^{p_2} provided $p_2 \leq 2p_1$, this proves the first assertion of the theorem.

Now suppose $2p_1 > p_2$. Let $(A_m)_{m \in N}$ be a sequence of infinite subsets of N satisfying $A_m \cap A_{m'} = \emptyset$ if $m \neq m'$.

We denote by S_m the set of all $x \in l^{p_2}$ such that $x \cdot e_n = 0$ for every $n \notin A_m$; S_m is an ideal of l^{p_2} . Let q be any algebra seminorm defined on l^{p_2} , and let (x_m) and (y_m) be two sequences of nonzero element of l^{p_2} satisfying $x_m \in S_m, y_m \in S_m$ ($m \in N$). For $m' \neq m$ we have $x_m \cdot x_{m'} = x_m \cdot y_{m'} = 0$.

By the version of Bade and Curtis main boundedness theorem given in [2, Theorem 3-1] we have

$$\sup_{m \in N} \frac{q(x_m y_m)}{\|x_m\|_{p_2} \|y_m\|_{p_2}} < +\infty$$

(in fact, as pointed out by A. M. Sinclair in [14], the condition $x_m \cdot x_{m'} = 0$ for $m \neq m'$ is not necessary).

So one can find a positive integer m_0 and a positive real K satisfying, for every $m > m_0$,

$$q(xy) \leq K \|x\|_{p_2} \|y\|_{p_2} \quad (x, y \in S_m). \quad (1)$$

Now suppose $l^{p_1} \subseteq \text{Ker } q$, and fix $m > m_0$. As $p_2 > 2p_1$, there exists $z \in S_m$ such that $z^2 \notin l^{p_1}$. We have

$$\lim_{n \rightarrow \infty} \|z - z f_n\|_{p_2} = 0 \quad \text{and} \quad z^2 - f_n z^2 = (z - f_n z)^2.$$

As $z^2 \cdot f_n$ belongs to l^{p_1} for every n , we obtain, using (1): $q(z^2) = 0$.

So l^{p_1} cannot be the kernel of an algebra seminorm defined on l^{p_2} , and the quotient algebra l^{p_2}/l^{p_1} is not normable with an algebra norm.

I must thank the referee who suggested to add to Theorem 3-1 that l^{p_2}/l^{p_1} is not normable if $p_2 > 2p_1$.

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