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EQUIVALENCES GENERATED BY FAMILIES OF BOREL SETS

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ABSTRACT. The equivalence relation on the reals generated by a family of $\aleph_\alpha$ Borel sets has either $< \aleph_\alpha$ or else exactly $2^{\aleph_0}$ equivalence classes.

As is usual in modern set theory, we identify an ordinal with the set of its predecessors, and a cardinal with the first ordinal of that cardinality. Thus $2 = \{0, 1\}$, $\omega = \{0, 1, 2, \dots\}$; while $\aleph_0 = \omega$, $\aleph_1 =$ the first uncountable ordinal ω_1, etc. If α, β are ordinals,

$$^{\beta}\alpha = \{f: f \text{ is a function } \& \operatorname{dom} f = \beta \ \& \operatorname{range} f \subseteq \alpha\};$$

while

$$^{\underset{\sim}{\beta}}\alpha = \bigcup_{\gamma<\beta} {}^{\gamma}\alpha.$$

If $f \in {}^{\beta}\alpha$ and $\gamma < \beta$, then $f|\gamma$ is the restriction of f to γ; while if $\delta < \alpha, f * \delta$ is the element g of ${}^{(\beta+1)}\alpha$ with $g|\beta = f$ and $g(\beta) = \delta$.

Let X be an uncountable Polish space (separable topological space admitting a complete metric), e.g. the reals. A family $\mathbb{S}$ of subsets of X generates an equivalence relation $E(\mathbb{S})$ on X defined by

$$xE(\mathbb{S})y \leftrightarrow \forall S \in \mathbb{S}(x \in S \leftrightarrow y \in S).$$

Let κ be an infinite cardinal. A subset $S \subseteq X$ is called *κ-Souslin* if S can be represented in the form

$$S = \bigcup_{f\in {}^{\omega}\kappa} \bigcap_{n\in\omega} C_{f|n},$$

where for each $s \in {}^{\underset{\sim}{\omega}}\kappa$, $C_s \subseteq X$ is closed. S is *co-κ-Souslin* if $X - S$ is κ-Souslin, and *bi-κ-Souslin* if both κ-Souslin and co-κ-Souslin. Thus the ω-Souslin sets are just the analytic (Σ^1_1) sets; the co-ω-Souslin sets are the $CA(\Pi^1_1)$ sets; and by a classical theorem of Souslin (see [3]) the bi-ω-Souslin sets are the Borel sets.

An equivalence relation E on X is said to have *perfectly many classes* if there is a perfect (closed, dense-in-itself) $P \subseteq X$ such that no two (distinct) elements of P are E-equivalent. Since any perfect subset of X has cardinality $2^{\aleph_0}$, this implies E has $2^{\aleph_0}$ classes. Note that if $\mathbb{S}$, $\mathbb{T}$ are families of subsets of

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X with $\mathfrak{T} \subseteq \mathfrak{S}$, then $E(\mathfrak{S}) \subseteq E(\mathfrak{T})$ (as subsets of X^2), and, hence, the number of $E(\mathfrak{S})$ classes can be no less than the number of $E(\mathfrak{T})$ classes, and the former has perfectly many classes if the latter does.

THEOREM. *Let X be an uncountable Polish space, κ an infinite cardinal, $\mathfrak{S}$ a family of κ many bi-κ-Souslin subsets of X. Then if the equivalence relation $E(\mathfrak{S})$ generated by $\mathfrak{S}$ has more than κ equivalence classes, there exists a countable $\mathfrak{T} \subseteq \mathfrak{S}$ such that $E(\mathfrak{T})$ has perfectly many classes.*

PROOF. Fix a complete metric ρ on X compatible with its topology. Enumerate $\mathfrak{S} = \{S^\alpha: \alpha < \kappa\}$. For each $\alpha < \kappa$ fix families ${}^iC_s^\alpha$ of closed subsets of X for $i = 0, 1$ and $s \in {}^{\underset{\smile}{\omega}}\kappa$, such that:

$$S^\alpha = \bigcup_{f \in {}^\omega\kappa} \bigcap_{n \in \omega} {}^0C^\alpha_{f|n}, \quad X - S^\alpha = \bigcup_{f \in {}^\omega\kappa} \bigcap_{n \in \omega} {}^1C^\alpha_{f|n}.$$

We may choose these families to be nested, so $s \subseteq t$ implies ${}^iC_t^\alpha \subseteq {}^iC_s^\alpha$; and we may choose them so that for $n \in \omega$ and $s \in {}^n\kappa$, the ρ-diameter of ${}^iC_s^\alpha$ is less than 2^{-n}. For $\alpha < \kappa$, $i = 0, 1$, and $s \in {}^{\underset{\smile}{\omega}}\kappa$, set

$${}^iS_s^\alpha = \bigcup_{\substack{f \in {}^\omega\kappa \\ s \subseteq f}} \bigcap_{n \in \omega} {}^iC^\alpha_{f|n} \subseteq {}^iC_s^\alpha.$$

Assume $E(\mathfrak{S})$ has $> \kappa$ classes, and let $Z \subseteq X$ be a set of κ^+ pairwise $E(\mathfrak{S})$-inequivalent elements.

We will define for every $l \in \omega$, $\sigma \in {}^l2$, an ordinal $\alpha(\sigma) < \kappa$ and elements $s(\sigma, k)$ of ${}^l\kappa$ for $k \leqslant l$, so that setting

$$T_\sigma = \bigcap_{k<l} {}^{\sigma(k)}S^{\alpha(\sigma|k)}_{s(\sigma,k)} \subseteq \bigcap_{k<l} {}^{\sigma(k)}C^{\alpha(\sigma|k)}_{s(\sigma,k)}, \tag{1}$$

we have $\mathrm{card}(Z \cap T_\sigma) = \kappa^+$. We will also arrange matters so that $\sigma \subseteq \tau$ implies $s(\sigma, k) \subseteq s(\tau, k)$ for all relevant k. We proceed by induction. Suppose then that $l \in \omega$, $\sigma \in {}^l2$, and suppose that for all $k < l$, $\alpha(\sigma|k)$ and $s(\sigma, k)$ have been defined and satisfy the conditions above.

In particular, $\mathrm{card}(Z \cap T_\sigma) = \kappa^+$. We claim this assumption implies that there exists an $\alpha < \kappa$ such that both $Z \cap T_\sigma \cap S^\alpha$ and $(Z \cap T_\sigma) - S^\alpha$ have cardinality κ^+. For suppose the opposite, and setting, for each $\alpha < \kappa$, M^α = whichever of $Z \cap T_\sigma \cap S^\alpha$ or $(Z \cap T_\sigma) - S^\alpha$ has cardinality $\leqslant \kappa$, we would find that all elements of $Z - \bigcup_{\alpha<\kappa} M^\alpha$ would be $E(\mathfrak{S})$-equivalent, hence that there could be only one such element, hence that card $Z = \kappa$, a contradiction! Let $\alpha(\sigma)$ be the least α with $\mathrm{card}(Z \cap T_\sigma \cap S^\alpha) = \mathrm{card}((Z \cap T_\sigma) - S^\alpha) = \kappa^+$. Now $Z \cap T_\sigma \cap S^{\alpha(\sigma)}$ is contained in

$$\bigcap_{k<l} {}^{\sigma(k)}S^{\alpha(\sigma|k)}_{s(\sigma,k)} \cap S^{\alpha(\sigma)} = \bigcap_{k<l} \bigcup_{\nu<\kappa} {}^{\sigma(k)}S^{\alpha(\sigma|k)}_{s(\sigma,k)*\nu} \cap \bigcup_{s \in {}^{(l+1)}\kappa} {}^0S_s^{\alpha(\sigma)}.$$

So there exist $\nu_0, \nu_1, \ldots, \nu_{(l-1)}$ and s such that setting $s(\sigma * 0, k) = s(\sigma, k) * \nu_k$ for $k < l$, and $s(\sigma * 0, l) = s$, and defining $T_{\sigma * 0}$ as *per* (1) above, we still have $\mathrm{card}(Z \cap T_{\sigma * 0}) = \kappa^+$. The $s(\sigma * 1, k)$ for $k \leqslant l$ are similarly defined.

For $g \in {}^\omega\omega$, $\{\overline{T}_{g|n}: n \in \omega\}$ forms a nested sequence of nonempty closed sets with ρ-diameters converging to 0. Hence this family intersects in a point $x_g \in X$. If $g(m) = 0$, then x_g belongs to

$$\bigcap_{n>m} {}^0C^{\alpha(g|m)}_{s(g|n,m)} \subseteq S^{\alpha(g|m)}.$$

Similarly, if $g(m) = 1$, then $x_g \notin S^{\alpha(g|m)}$. Thus if g, h are two (distinct) elements of ${}^\omega\omega$, x_g, x_h are $E(\mathbb{S})$-inequivalent, and incidentally $x_g \neq x_h$. Thus $A = \bigcup_{g \in {}^\omega\omega} \bigcap_{n\in\omega} \overline{T}_{g|n} = \{x_g: g \in {}^\omega\omega\}$ is an uncountable analytic set, and hence contains a perfect subset P. Moreover, setting $\mathfrak{T} = \{S^{\alpha(\sigma)}: \sigma \in {}^{\underset{\smile}{\omega}}2\}$, any two elements of P are $E(\mathfrak{T})$-inequivalent, proving the theorem. $\square$

Corollary 1. *Let X be a Polish space, κ an infinite cardinal. Then any equivalence relation on X which is an intersection of κ CA equivalences has either $\leqslant \kappa$ or else perfectly many equivalence classes.*

Proof. We use a deep theorem of Silver **[4]**: *Any $CA(\mathbf{\Pi}^1_1)$ equivalence relation on a Polish space X has either countably many or else perfectly many equivalence classes*. Now let E be an equivalence on a Polish space X of form $\bigcap_{\alpha<\kappa} E_\alpha$ where the E_α are CA equivalences. If any E_α has perfectly many classes, so does E. If each E_α has only countably many classes $\{S_{\alpha,n}: n < N_\alpha\}$, $N_\alpha \leqslant \omega$, then each of these $S_{\alpha,n}$ is both CA (since E_α is CA) and analytic (being the complement of $\bigcup_{m\neq n} S_{\alpha,m}$) and hence is Borel. Thus in this case $E = E(\mathbb{S})$ where $\mathbb{S} = \{S_{\alpha,n}: \alpha < \kappa, n < N_\alpha\}$ is a family of κ Borel sets. Thus *any intersection of κ CA equivalences either has perfectly many classes or else is generated by a family of κ Borel sets*. Corollary 1 is immediate. This corollary answers a question of J. Steel. $\square$

The referee has informed us that V. Harnik and M. Makkai **[5]** have obtained Corollary 1 (for X = Baire space) by a model-theoretic argument. The Theorem has somewhat more scope than this corollary, implying e.g. that if $\mathbb{S}$ is a family of $\aleph_1$ analytic sets, $E(\mathbb{S})$ has $\leqslant \aleph_1$ or perfectly many classes.

Corollary 2. *Any analytic equivalence relation on a Polish space X has either $\leqslant \omega_1$ or else perfectly many classes.*

Proof. Elsewhere **[2]** we have shown: *Any analytic equivalence relation on a Polish space X is an intersection of ω_1 Borel equivalences*. Corollary 2 is then immediate. Actually in **[2]** we establish more: A $CPCA(\mathbf{\Pi}^1_2)$ equivalence of the special form $xEy \leftrightarrow \forall z \in X\,(x, y, z) \in D$, where $D \subseteq X^3$ is analytic, and for each fixed z, $\{(x,y): (x,y,z) \in D\}$ is an equivalence relation, is an intersection of ω_1 CA equivalences. So the cardinal estimates on the number of classes in Corollary 2 apply to such special $CPCA$ equivalences, too. Corollary 2 was the main result of our thesis **[1]**. It answers a question of H. Friedman. $\square$

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