ON THE GENUS OF GRAPHS WITH LICK-WHITE NUMBER k

JOHN MITCHEM

ABSTRACT. A graph is called n -degenerate if each of its subgraphs has a vertex of degree at most n . For each n the Lick-White number of graph G is the fewest number of sets into which $V(G)$ can be partitioned such that each set induces an n -degenerate graph. An upper bound is obtained for the Lick-White number of graphs with given clique number. A number of estimates are derived for the number of vertices in triangle-free graphs with prescribed Lick-White number. These results are used to give lower bounds on the genus of such graphs.

In 1948, B. Descartes [8] proved the existence of triangle-free graphs with chromatic number 4. Since then, relationships between the chromatic number and girth have been investigated by a number of authors. Perhaps the most startling result is that proved, using different methods, by Erdős [9] and Lovász [16].

THEOREM A (ERDŐS AND LOVÁSZ). *For every pair of integers k, t , where $k \geq 2$, there is a graph of G with $\chi(G) = k$ and $g(G) \geq t$, where $\chi(G)$ is the chromatic number and $g(G)$ is the girth of G .*

More recently Lick and White [14] and [15] used the concept of n -degenerate graph in order to generalize chromatic number. For nonnegative integer n , a graph is called n -degenerate if each of its subgraphs has a vertex of degree at most n . For each n the Lick-White number of graph G is denoted by $\chi_n(G)$ and is defined as the least number of sets into which $V(G)$ can be partitioned such that each set induces an n -degenerate graph. Other papers which include results on Lick-White numbers include Simões-Pereira [21], [22], and [23], Cook [7], Jones [12] and Mitchem [18]. The chromatic number and point-arboricity of G are respectively $\chi_0(G)$ and $\chi_1(G)$. For definitions and notation not defined in the papers see Harary [11].

A natural problem is to attempt to generalize results on girth and chromatic number to Lick-White numbers and girth. In [19] Mitchem constructed graphs with $g(G) = 4$ and $f_n(G)$ arbitrarily large, where $f_n(G)$ is different from but analogous to $\chi_n(G)$. By very slightly modifying that work one obtains graphs with $g(G) = 4$ and $\chi_n(G)$ arbitrarily large. Simões-Pereira [21] constructed much smaller graphs with the same property. In [7] Cook proves,

Received by the editors July 23, 1976 and, in revised form, April 15, 1977.

AMS (MOS) subject classifications (1970). Primary 05C15; Secondary 05C10.

Key words and phrases. n -degenerate graph, Lick-White number, genus.

© American Mathematical Society 1978

using the methods of Erdős:

THEOREM B. *For every integer triple n, k, t ; $n \geq 0, k \geq 2, t \geq 1$, there is a graph G with $\chi_n(G) = k$ and $g(G) \geq t$.*

Cook's proof, unlike those of Simões-Pereira and Mitchem, shows existence, but does not actually construct the required graph. In this paper we use Theorem A to give a very short proof of Theorem B. Also, we will give a new lower bound, Theorem 9, which improves a result of Cook, on the genus of a triangle-free graph G with $\chi_n(G) = k$. Catlin [4] recently gave a significant improvement of Brooks' Theorem for graphs with small clique number. In Theorem 2 we generalize Catlin's Theorem using Mitchem's [18] new generalization of Brooks' Theorem. Theorem 2 is used in the proof of Theorem 9. We also improve certain bounds given by Chvátal [5] and Kronk [13].

Borodin and Kostochka [3] as well as Bollobás and Thomason [1] have also proved Theorem 2. The former uses Borodin's version [2] of Theorem E. Theorem B is further improved in [1] where the existence is proved of a graph G which is uniquely partitionable into k n -degenerate graphs and $g(G) \geq t$.

The maximum (minimum) degree of G is denoted by $\Delta(G)$ ($\delta(G)$). If $U \subset V(G)$, then $\langle U \rangle$ denotes the subgraph induced by U . The greatest integer function and least integer function are denoted respectively by $[x]$ and $\{x\}$. In order to give our proof of Theorem B, we need the following elementary results.

THEOREM C (LICK AND WHITE [14]). *A graph G is n -degenerate iff $V(G)$ can be ordered $v_1, v_2, \dots, v_p$ such that in graph $\langle \{v_i, v_{i+1}, \dots, v_p\} \rangle$, $\deg v_i \leq n$.*

THEOREM 1. *For every $n \geq 1$, $\chi_0(G)/(n+1) \leq \chi_n(G) \leq \chi_0(G)$.*

PROOF. The second inequality is immediate. For the first inequality, let $\chi_n(G) = h$. Thus there is a partition $W_1, \dots, W_h$ of $V(G)$ such that each $\langle W_i \rangle$ is n -degenerate. It follows from Theorem C that $\chi_0(\langle W_i \rangle) \leq n+1$ for each i . Hence, $\chi_0(G) \leq (n+1)(\chi_n(G))$.

PROOF OF THEOREM B. Consider $k' = (n+1)k$. By Theorem A there is a graph G with $g(G) \geq t$ and $\chi_0(G) = k'$. By Theorem 1, $\chi_n(G) \geq k$. Now remove sufficient points from G until G' is obtained such that $\chi_n(G') = k$ and $g(G') \geq t$.

As did Catlin, we will use the following theorem.

THEOREM D (LOVÁSZ [17]). *For any positive integer a , if $h_1, \dots, h_a$ are nonnegative integers such that $h_1 + \dots + h_a = \Delta(G) - a + 1$, then there is a partition $V_1, \dots, V_a$ of $V(G)$ such that $\Delta(\langle V_i \rangle) \leq h_i$ for $i = 1, 2, \dots, a$.*

THEOREM E (MITCHEM [18]). *For $n \geq 0$, let G be a connected graph other than*

- (i) *A complete graph with $t(n + 1) + 1$ vertices, $n \geq 0, t \geq 0$.*
 - (ii) *An $(n + 1)$ -regular graph, $n \geq 0$.*
 - (iii) *An odd cycle, $n = 0$.*
- Then $\chi_n(G) \leq \{\Delta(G)/(n + 1)\}$.

Theorem E can be viewed as a natural analogue to Brooks' Theorem and will also be used to prove the next theorem, which shows that if a graph has small clique number then the bound given in Theorem E can be improved.

THEOREM 2. *Suppose G is a graph with clique number s , and let n be a nonnegative integer. Also let r be the smallest integer of the form $t(n + 1)$, $t \geq 2$, such that $r \geq \max\{3, s\}$. Then $\chi_n(G) \leq \{(\Delta(G) + 1 - m)/(n + 1)\}$ where $m = [(\Delta(G) + 1)/(r + 1)]$.*

PROOF. By Theorem E, if $m = 0$ or 1 , Theorem 2 is true. Assume $m \geq 2$ and let $h_i = r$ for $i = 1, \dots, m - 1$ and

$$h_m = \Delta(G) + r + 1 - (r + 1)[(\Delta(G) + 1)/(r + 1)] \geq r.$$

By Theorem D, there exists a partition $V_1, \dots, V_m$ of $V(G)$ such that $\Delta(\langle V_i \rangle) \leq r$ for $i = 1, \dots, m - 1$ and $\Delta(\langle V_m \rangle) \leq \Delta(G) + (r + 1)(1 - m)$. Since G has clique number $s \leq r$, no $\langle V_i \rangle$ contains K_{r+1} . According to Theorem E,

$$\chi_n(\langle V_i \rangle) \leq \{r/(n + 1)\} = t \quad \text{for } i = 1, \dots, m - 1,$$

and

$$\chi_n(\langle V_m \rangle) \leq \{(\Delta(G) + (r + 1)(1 - m))/(n + 1)\}.$$

It follows that

$$\begin{aligned} \chi_n(G) &\leq \chi_n(\langle V_1 \rangle) + \dots + \chi_n(\langle V_m \rangle) \\ &= (m - 1)t + \{(\Delta(G) + 1 - t(n + 1)(m - 1) - m)/(n + 1)\} \\ &= \{(\Delta(G) + 1 - m)/(n + 1)\}. \end{aligned}$$

COROLLARY F (CATLIN). *If graph G has clique number s , then $\chi_0(G) \leq \Delta(G) + 1 - [(\Delta(G) + 1)/(\max\{3, s\} + 1)]$.*

It should be noted that Grünbaum [10] asked that for every $k \geq 4$ does there exist a triangle-free graph G with $\chi_0(G) = k = \Delta(G)$. Chvátal [6] has found the smallest such graph with $k = 4$. Catlin's Theorem implies no such graph exists with $k \geq 7$.

Let $f_n(k)$ be the minimum number of vertices in a triangle-free graph G with $\chi_n(G) = k$. It is transparent that $f_0(2) = 2, f_0(3) = 5$. Chvátal [5] showed that $f_0(4) = 11$, and that the so-called Mycielski graph [20] is the only graph which obtains this minimum.

THEOREM 3. *For $n \geq 0, f_n(2) = 2(n + 1)$.*

PROOF. The graph $G = K(n + 1, n + 1)$ shows that $f_n(2) \leq 2(n + 1)$.

Assume that H is the smallest graph which has $\chi_n(H) = 2$. Thus $\delta(H) \geq n + 1$ and $2q \geq (n + 1)p$ where q and p are the number of edges and vertices of G , respectively. However, by Turan's Theorem [24] any triangle-free graph has $2q < p^2/2$, and thus $p \geq 2(n + 1)$.

We now give lower and upper bounds on $f_n(3)$. The upper bound is proved by constructing the smallest known triangle-free graph G with $\chi_n(G) = 3$.

THEOREM 4. For $n \geq 0$, $f_n(3) \leq (2n + 1)5$.

PROOF. Let H_1 be the 5-cycle v_1, v_2, v_3, v_4, v_5 and $\bar{H}_2 = K_{2n+1}$. Define G as the composition $H_1[H_2]$. (See Harary [11].) Now $g(G) = 4$. For $i = 1, 2, 3, 4, 5$ let V_i be the set of vertices of G with first coordinate v_i . The partition $V_1 \cup V_3, V_2 \cup V_4, V_5$ shows that $\chi_n(G) \leq 3$. Suppose that $V(G)$ can be partitioned into n -degenerate sets R and S . At least $(n + 1)$ points of V_1 are in one of the sets, say R . Thus at least $(n + 1)$ points of V_2 are in S . It follows that at least $(n + 1)$ points of V_4 are also in S . This implies that S contains at most n vertices of V_5 , as does R , a contradiction.

Before proving Theorem 5, we state a needed definition and theorem. Graph G is called (n, k) -critical if $\chi_n(G) = k$ and for every proper subgraph H of G , $\chi_n(H) < k$.

THEOREM G (LICK AND WHITE). If G is (n, k) -critical, then $\delta(G) \geq (n + 1)(k - 1)$.

THEOREM 5. For $n \geq 1$, $f_n(3) \geq 6n + 7$.

PROOF. Let G be a triangle-free graph of minimum order with $\chi_n(G) = 3$. By Theorems G and E we have that $\delta(G) \geq 2n + 2$ and $\Delta(G) \geq 2n + 3$. Let v be a point of maximum degree with neighbor set $W = \{w_1, \dots, w_t\}$, $t \geq 2n + 3$. Since $g(G) > 3$, no two w_i are adjacent. Let $B = V(G) - W - \{v\}$. Since $\chi_n(\langle B \rangle) \geq 2$, $\langle B \rangle$ contains a path with at least $2n$ vertices. Suppose there is a path $P = u_1, u_2, \dots, u_s$, $s \leq 2n$, such that $\langle B \rangle - P$ is n -degenerate. Then $(B - V(P)) \cup \{v\}$, $V(P) \cup W$ is a partition of $V(G)$ into two n -degenerate sets, a contradiction. It follows that there is a path $Q = v_1, \dots, v_{2n}$ in $\langle B \rangle$ such that $\chi_n(\langle B \rangle - Q) = 2$. Thus there is a subgraph H of $\langle B \rangle - Q$ with $\delta(H) \geq n + 1$. Since H is triangle-free, it has at least $2n + 2$ points. This implies that the order p of G is at least $6n + 6$. Suppose $p = 6n + 6$, thus $H = \langle B \rangle - Q = K(n + 1, n + 1)$, so that $V(H) = X \cup Y$ where every vertex of X is adjacent with every vertex of Y . Any vertex of $W \cup V(Q)$ is adjacent with elements of only one of X and Y . Also, each element of W is adjacent to at most n elements of $V(Q)$. Let $W_x = \{w_i: w_i \text{ is adjacent with no element of } X\}$ and $W_y = \{w_i: w_i \text{ is adjacent with no element of } Y\}$. Then $X_1 = X \cup W_x \cup \{v_1, v_2, \dots, v_n\}$, $X_2 = Y \cup W_y \cup \{v_{n+1}, \dots, v_{2n}\} \cup \{v\}$ yields a partition of $V(G)$ into 2 n -degenerate sets. This contradiction implies that $f_n(3) \geq 6n + 7$.

THEOREM 6. For $n \geq 0$, $k \geq 4$, $f_n(k) \geq f_n(k - 1) + kn + k + n - 1 +$

k/t , where $t = 3$ if $n = 0$, and $t = 2$ otherwise.

PROOF. Let G have $\chi_n(G) = k$, $\deg v = \Delta(G)$, and W be the set of points adjacent with vertex v of degree $\Delta(G)$ in G . Theorem 2 implies that $|W| \geq kn + k - n - 2 + k/t$. Now $G - W - \{v\}$ contains an $(n, k - 1)$ -critical graph H with $\delta(H) \geq (k - 2)(n + 1)$ and thus H contains a path P with $2n$ points. Since H is triangle-free, each point of W is adjacent with at most n points of P and thus $\langle V(P) \cup W \rangle$ is n -degenerate. It follows that $\chi_n(G - P - W - \{v\}) \geq k - 1$. Thus $|G| \geq f_n(k - 1) + kn + k - n - 2 + k/t + 2n + 1$.

COROLLARY 7. For $n \geq 1$, $k \geq 3$, and for $n = 0$, $k \geq 5$, $f_n(k) \geq 4 - 3n + (k^2 + k - 12)/2t - k/2 + 3nk/2 + k^2(n + 1)/2$.

PROOF. For $n \geq 1$, $k \geq 3$, the corollary follows immediately from the difference equation given in Theorem 6 together with Theorem 5. For $n = 0$, $k \geq 5$ the corollary is implied by using $f_0(5) \geq 17$ from Chvátal [5] as our boundary condition and Theorem 6.

COROLLARY 8. For $k \geq 5$, $f_0(k) \geq 2k^2/3 - k/3 + 2$.

In [5] Chvátal shows that $f_0(k) \geq (k + 2)(k + 1)/2 - 4$ for $k \geq 4$. For $k \geq 7$, the bound given in Corollary 8 is better. Although there exist graphs with no triangles and arbitrarily high Lick-White numbers, Cook by Theorem H showed that such graphs necessarily have large genus.

THEOREM H (COOK [1]). Let G be a graph of positive genus, $\gamma(G)$, with $g(G) \geq 4$. Then $\chi_n(G) \leq 1 + [(2 + 2\sqrt{\gamma(G)})/(n + 1)]$.

Theorem H can be viewed as a lower bound for $\gamma(G)$ in terms of n and $\chi_n(G)$. Our final theorem improves Theorem H.

THEOREM 9. Let G be a triangle-free graph with $\chi_n(G) = k$ where $k \geq 3$ if $n \geq 1$, and $k \geq 5$ if $n = 0$.

$$\begin{aligned} 8\gamma(G) \geq & (kn + k - n - 5) \\ & (4 - 3n + (3nk + k^2n + k^2 - k)/2 + (k^2 + k - 12)/2t) \\ & + k/t + 7 \end{aligned}$$

where $t = 3$ if $n = 0$ and $t = 2$ if $n \geq 1$.

PROOF. Since every graph with $\chi_n(G) = k$ contains a (n, k) -critical subgraph, it suffices to prove Theorem 9 for (n, k) -critical graphs. Let G have p points and q lines. From Theorems G and 2 we have

$$2q \geq (k - 1)(n + 1)p + k/t - 1.$$

However, from the generalized Euler formula

$$2q \leq 8\gamma(G) + 4p - 8$$

it follows that

$$8\gamma(G) \geq [(k-1)(n+1) - 4]p + k/t + 7.$$

The result follows by Corollary 7.

In [13] Kronk obtained a lower bound for $\gamma(G)$ in terms of $\chi_n(G) = k$ and $g(G)$, if G is triangle-free. His bound is quadratic in k . Thus, for each value of $g(G) \geq 4$, there is a value k_0 such that if $\chi_0(G) \geq k_0$, then Theorem 9 gives a better bound than Kronk's.

REFERENCES

1. B. Bollobás and A. G. Thomason, *Uniquely partitionable graphs* (to appear).
2. O. V. Borodin, *On decomposition of graphs into degenerate subgraphs*, Diskret. Analiz **28** (1976). (Russian)
3. O. V. Borodin and A. V. Kostochka, *On an upper bound of the graph's chromatic number depending on graph's degree and density*, Novosibirsk, Siberia (preprint).
4. P. Catlin, *A bound on the chromatic number of graphs* (preprint).
5. V. Chvátal, *The minimality of the Mycielski graph*, Lecture Notes in Math., vol. 406, Springer-Verlag, Berlin and New York, 1974, pp. 243–246.
6. ———, *The smallest triangle-free 4-chromatic, 4-regular graph*, J. Combinatorial Theory **9** (1970), 93–94.
7. R. J. Cook, *Point partition numbers and girth*, Proc. Amer. Math. Soc. **49** (1975), 510–514.
8. B. Descartes, *A three-colour problem*, Eureka **10** (1948), 24.
9. P. Erdős, *Graph theory and probability*, Canad. J. Math. **11** (1959), 34–38.
10. B. Grünbaum, *A problem in graph coloring*, Amer. Math. Monthly **77** (1970), 1080–1092.
11. F. Harary, *Graph theory*, Addison-Wesley, New York, 1969.
12. R. P. Jones, *Hereditary properties and P-chromatic numbers*, Combinatorics, London Math. Soc. Lecture Note Ser., vol. 13, Cambridge Univ. Press, London, 1974, pp. 83–88.
13. H. V. Kronk, *The chromatic number of triangle-free graphs*, Graph Theory and Applications, Lecture Notes in Math., vol. 303, Springer-Verlag, Berlin and New York, 1972, pp. 179–181.
14. D. R. Lick and A. T. White, *k-degenerate graphs*, Canad. J. Math. **12** (1970), 1082–1096.
15. ———, *The point-partition numbers of closed 2-manifolds*, J. London Math. Soc. (2) **4** (1972), 577–587.
16. L. Lovász, *On chromatic number of finite set-systems*, Acta Math. Acad. Sci. Hungar. **19** (1968), 59–67.
17. ———, *On decomposition of graphs*, Studia Sci. Math. Hungar. **1** (1966), 237–238.
18. J. Mitchem, *An extension of Brooks' theorem to n-degenerate graphs*, Discrete Math. **17** (1977), 291–298.
19. ———, *On extremal partitions of graphs*, Ph. D. Dissertation, Western Michigan Univ., 1970, pp. 103–109.
20. J. Mycielski, *Sur le coloriage des graphes*, Colloq., Math. **3** (1955), 161–162.
21. J. M. S. Simões-Pereira, *A note on graphs with prescribed clique and point-partition numbers*, J. Combinatorial Theory Ser. B **14** (1973), 256–258.
22. ———, *Joins of n-degenerate graphs and uniquely (m, n)-partitionable graphs*, J. Combinatorial Theory Ser. B **21** (1976), 21–29.
23. ———, *On graphs uniquely partitionable into n-degenerate subgraphs*, Coll. Math. Soc. Janos Bolyai, 10 Infinite and Finite Sets, Keszthely (Hungary), 1973.
24. P. Turan, *On the theory of graphs*, Colloq. Math. **3** (1954), 19–30.

DEPARTMENT OF MATHEMATICS, SAN JOSE STATE UNIVERSITY, SAN JOSE, CALIFORNIA 95192