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LOCAL COMPACTNESS AND HEWITT REALCOMPACTIFICATIONS OF PRODUCTS

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ABSTRACT. In this note we prove McArthur's conjecture [6]: If card X is nonmeasurable and if $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each space Y, then X is locally compact. Consequently, we can completely characterize the class of all spaces X such that for each space Y, $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds.

1. Introduction. All spaces considered in this note will be completely regular Hausdorff. For a space X, υX denotes the Hewitt realcompactification of X, and the symbolism $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ means that $X \times Y$ is C-embedded in $\upsilon X \times \upsilon Y$. Following [6], let $\mathfrak{R}$ denote the class of all spaces X such that for each space Y, $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds. It is known that a locally compact realcompact space of nonmeasurable cardinal is a member of $\mathfrak{R}$ and that every member of $\mathfrak{R}$ is realcompact (Comfort [1, Corollary 2.2], McArthur [6, Theorem 5.2]). In [6], McArthur conjectured that if card X is nonmeasurable and X is a member of $\mathfrak{R}$, then X is locally compact. The main purpose of this note is to establish his conjecture positively. More precisely, we can prove the following theorems. The implication (a) $\rightarrow$ (b) of Theorem 1 was proved by Comfort [1].

THEOREM 1. *For a space X of nonmeasurable cardinal the following conditions are equivalent*:

(a) *X is locally compact.*

(b) *$X \times Y$ is C-embedded in $X \times \upsilon Y$ for each space Y.*

THEOREM 2. *For a space X of nonmeasurable cardinal the following conditions are equivalent*:

(a) *X is locally pseudocompact.*

(b) *$X \times Y$ is C-embedded in $X \times \upsilon Y$ for each k-space Y.*

We remark that, in Theorems 1 and 2, the assumption "card X is nonmeasurable" is useful only for the implication (a) $\rightarrow$ (b). Combining these theorems with the results of Comfort and McArthur, quoted above, and Hušek [4, Theorem 3], we have the following theorems.

THEOREM 3. *For a space X the following conditions are equivalent*:

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(a) *X is locally compact, realcompact and* card *X is nonmeasurable.*
(b) $v(X \times Y) = vX \times vY$ *holds for each space* Y.

THEOREM 4. *For a space X the following conditions are equivalent*:
(a) vX *is locally compact and* card X *is nonmeasurable.*
(b) $v(X \times Y) = vX \times vY$ *holds for each k-space* Y.

For the notions of locally pseudocompact spaces and k-spaces see [**1**]. For an ordinal α, we denote by $W(\alpha)$ the set of all ordinals less than α topologized with order topology, and by ω_0 the first infinite ordinal. Other terms can be found in [**3**].

2. Proofs of theorems.

PROOF OF THEOREM 1. (a) $\to$ (b). This is the result of Comfort [**1**, Theorem 2.1]. (b) $\to$ (a). Suppose, on the contrary, that X is not locally compact at $x_0 \in X$. Let $\{G_\lambda | \lambda \in \Lambda\}$ be a neighborhood base at x_0 in X. Then, for each $\lambda \in \Lambda$, $\mathrm{cl}_X G_\lambda$ is not compact, and thus there exists a point $x_\lambda \in \mathrm{cl}_{\beta X} G_\lambda \cap (\beta X - X)$, where βX is the Stone-Čech compactification of X. For each $\lambda \in \Lambda$, let $\{G(\lambda, \sigma) | \sigma \in \Sigma_\lambda\}$ be a neighborhood base at x_λ in βX. For each $\sigma \in \Sigma_\lambda$, we can choose a point $x(\lambda, \sigma) \in X$ and an open set $H(\lambda, \sigma)$ in X such that $x(\lambda, \sigma) \in H(\lambda, \sigma) \subset G(\lambda, \sigma) \cap G_\lambda$. Let s_λ be an ideal point, and set $S_\lambda = \Sigma_\lambda \cup \{s_\lambda\}$, topologized as follows: Each point of Σ_λ is isolated and $\{J(\lambda, \sigma) | \sigma \in \Sigma_\lambda\}$ is a neighborhood base at s_λ, where $J(\lambda, \sigma) = \{s_\lambda\} \cup \{\tau \in \Sigma_\lambda | G(\lambda, \sigma) \supset G(\lambda, \tau)\}$. Let $\mathfrak{n}$ be a regular cardinal greater than $\sup\{\mathrm{card}\, \Sigma_\lambda | \lambda \in \Lambda\}$, and let ω_α be the initial ordinal of $\mathfrak{n}$. For each $\lambda \in \Lambda$, let

$$T_{\lambda(1)} = \{(\lambda(1), \gamma, \beta) | \gamma \leqslant \omega_\alpha, \beta \leqslant \omega_0\}$$

be the copy of $W(\omega_\alpha + 1) \times W(\omega_0 + 1)$, and let

$$T_{\lambda(2)} = \{(\lambda(2), \gamma, s) | \gamma \leqslant \omega_\alpha, s \in S_\lambda\}$$

be the copy of $W(\omega_\alpha + 1) \times S_\lambda$. By identifying a point $(\lambda(1), \gamma, \omega_0)$ with $(\lambda(2), \gamma, s_\lambda)$ for $\gamma \leqslant \omega_\alpha$, we have a quotient space T_λ and a quotient map f_λ: $T_{\lambda(1)} \oplus T_{\lambda(2)} \to T_\lambda$, where $A \oplus B$ denotes the topological sum of A and B. Let us set $Z = \bigoplus \{T_\lambda | \lambda \in \Lambda\}$, and let Y_0 be the quotient space obtained from Z be collapsing a set $\{f_\lambda((\lambda(1), \omega_\alpha, \beta)) | \lambda \in \Lambda\}$ to a single point $y(\beta) \in Y_0$ for $\beta \leqslant \omega_0$. Let g: $Z \to Y_0$ be the quotient map, and set $h_\lambda = g \circ f_\lambda$ for each $\lambda \in \Lambda$. Then $y(\omega_0) = h_\lambda((\lambda(2), \omega_\alpha, s_\lambda))$ for each $\lambda \in \Lambda$. Let us set $Y = Y_0 - \{y_0\}$, where $y_0 = y(\omega_0)$. We shall now prove that $Y_0 \subset vY$ by showing that Y is C-embedded in Y_0. Let ϕ be a real-valued continuous function on Y. For each $\lambda \in \Lambda$, by the same argument as in [**3**, 8.20], there is $\gamma_\lambda \in W(\omega_\alpha)$ such that $\theta_\lambda = \phi \circ (h_\lambda | h_\lambda^{-1}(Y))$ takes on the constant value t_λ on $\{(\lambda(1), \gamma, \omega_0) | \gamma_\lambda \leqslant \gamma < \omega_\alpha\} \cup \{(\lambda(2), \gamma, s_\lambda) | \gamma_\lambda \leqslant \gamma < \omega_\alpha\}$. Since

$$\theta_\lambda((\lambda(1), \omega_\alpha, \beta)) = \theta_\mu((\mu(1), \omega_\alpha, \beta))$$

for $\lambda, \mu \in \Lambda$ and for each $\beta < \omega_0$, we have $t_\lambda = t_\mu$ for $\lambda, \mu \in \Lambda$. Extend ϕ over Y_0 by setting $\phi(y_0) = t_\lambda$. Then it is easy to see that the extension ϕ is continuous. Thus Y is C-embedded in Y_0, and hence $Y_0 \subset \upsilon Y$. It remains to show that $X \times Y$ is not C-embedded in $X \times \upsilon Y$. For each $\lambda \in \Lambda$ and each $\sigma \in \Sigma_\lambda$, let us set

$$y(\lambda, \sigma) = h_\lambda\big((\lambda(2), \omega_\alpha, \sigma)\big),$$
$$K(\lambda, \sigma) = h_\lambda\big(\{(\lambda(2), \gamma, \sigma) | \gamma \leqslant \omega_\alpha\}\big).$$

And let us set

$$p(\lambda, \sigma) = \big(x(\lambda, \sigma), y(\lambda, \sigma)\big) \in X \times Y,$$
$$L(\lambda, \sigma) = H(\lambda, \sigma) \times K(\lambda, \sigma) \subset X \times Y,$$
$$\mathfrak{L} = \{L(\lambda, \sigma) | \lambda \in \Lambda, \sigma \in \Sigma_\lambda\}.$$

Then $L(\lambda, \sigma)$ is a neighborhood at $p(\lambda, \sigma)$ in $X \times Y$. Now we show that $\mathfrak{L}$ is discrete in $X \times Y$. To do this, let $p = (x, y) \in X \times Y$; then $y = h_\mu((\mu(i), \delta, t))$ for some $\mu \in \Lambda$, $i \in \{1, 2\}$, $\delta \leqslant \omega_\alpha$ and $t \in W(\omega_0 + 1) \oplus S_\mu$. If $t \in W(\omega_0 + 1)$ and $t < \omega_0$, then

$$V(y) = \cup \{h_\lambda(T_{\lambda(1)}) | \lambda \in \Lambda\} \cap Y$$

is a neighborhood at y in Y such that $V(y) \cap K(\lambda, \sigma) = \varnothing$ for each $\lambda \in \Lambda$ and each $\sigma \in \Sigma_\lambda$, and hence $X \times V(y)$ is a neighborhood at p which meets no member of $\mathfrak{L}$. If $t = \omega_0$ or s_μ, then there exist $\tau \in \Sigma_\mu$ and a neighborhood $V(x)$ at x such that $V(x) \cap G(\mu, \tau) = \varnothing$. If we set

$$V(y) = h_\mu\Big(\{(\mu(1), \gamma, \beta) | \gamma \leqslant \delta, \beta \leqslant \omega_0\}$$
$$\cup \{(\mu(2), \gamma, s) | \gamma \leqslant \delta, s \in J(\mu, \tau)\}\Big),$$

then $V(y)$ is a neighborhood at y in Y such that $V(x) \times V(y)$ meets no member of $\mathfrak{L}$. If $t \in \Sigma_\mu$, then $X \times K(\mu, t)$ is a neighborhood at p which meets only $L(\mu, t)$. Hence $\mathfrak{L}$ is discrete in $X \times Y$. For each $\lambda \in \Lambda$ and each $\sigma \in \Sigma_\lambda$, there is a real-valued continuous function $\psi_{(\lambda,\sigma)}$ on $X \times Y$ such that $\psi_{(\lambda,\sigma)}(p(\lambda, \sigma)) = 0$ and $\psi_{(\lambda,\sigma)}(q) = 1$ for each $q \in (X \times Y) - L(\lambda, \sigma)$. If we define a function ψ by

$$\psi(q) = \inf\{\psi_{(\lambda,\sigma)}(q) | \lambda \in \Lambda, \sigma \in \Sigma_\lambda\}, \qquad q \in X \times Y,$$

then ψ is continuous, since $\mathfrak{L}$ is discrete. For our purpose, it suffices to show that ψ admits no continuous extension to the point $p_0 = (x_0, y_0) \in X \times \upsilon Y$. Let U be a given neighborhood at p_0. There exist $\mu \in \Lambda$ and a neighborhood $V(y_0)$ at y_0 in Y_0 such that $p_0 \in G_\mu \times V(y_0) \subset U$. Then $y(\mu, \tau) \in V(y_0)$ for some $\tau \in \Sigma_\mu$, and hence $p(\mu, \tau) \in U$ and $\psi(p(\mu, \tau)) = 0$. On the other hand, $y(\beta)$ is in $V(y_0)$ for some $\beta < \omega_0$, and then $q = (x_0, y(\beta)) \in U$ and $\psi(q) = 1$. This shows that ψ does not extend continuously to p_0. Hence the proof is completed.

Before proving Theorem 2, we prove the implication (a) $\rightarrow$ (b) of Theorem 4, which slightly improves a theorem of Comfort [**1**, Theorem 2.4]. We denote by μX the topological completion of X (i.e., the completion of X with respect to its finest uniformity).

PROOF OF THEOREM 4. (a) $\to$ (b). Assume that υX is locally compact and card X is nonmeasurable. Let Y be a k-space. Then, by [**1**, Theorem 2.1], $\upsilon X \times Y$ is C-embedded in $\upsilon X \times \upsilon Y$. Since υX is locally compact, by [**5**, Theorem 1.5], we have $\upsilon X = \mu X$. Hence $\mu(X \times Y) = \mu X \times \mu Y$ holds by [**5**, Theorem 2.3], and so $X \times Y$ is C-embedded in $\mu X \times Y$ $(=\upsilon X \times Y)$. Thus we have $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$.

PROOF OF THEOREM 2. (a) $\to$ (b). Let X be a locally pseudocompact space of nonmeasurable cardinal and let Y be a k-space. Now it suffices to show that for each pseudocompact subset S of X, $S \times Y$ is C-embedded in $S \times \upsilon Y$. To see this, let S be a given pseudocompact subset of X, then we have $\upsilon S = \beta S$ by [**3**, 8A4]. Thus $\upsilon(S \times Y) = \upsilon S \times \upsilon Y$ holds by Theorem 4, (a) $\to$ (b) proved above, and hence $S \times Y$ is C-embedded in $S \times \upsilon Y$. (b) $\to$ (a). Suppose on the contrary that X is not locally pseudocompact at $x_0 \in X$. Let $\{G_\lambda | \lambda \in \Lambda\}$ be a neighborhood base at x_0. Then, for each $\lambda \in \Lambda$, $\mathrm{cl}_X G_\lambda$ is not pseudocompact, and thus we can find a countable decreasing family $\{G(\lambda, \sigma) | \sigma \in \Sigma_\lambda\}$ of open sets in X such that $\bigcap\{\mathrm{cl}_X G(\lambda, \sigma) | \sigma \in \Sigma_\lambda\} = \varnothing$ and $G(\lambda, \sigma) \subset G_\lambda$ for each $\sigma \in \Sigma_\lambda$. Let us set $H(\lambda, \sigma) = G(\lambda, \sigma)$, and choose a point $x(\lambda, \sigma) \in H(\lambda, \sigma)$. We construct Y_0 and Y quite similarly to the proof of Theorem 1. Examining the process, one sees that then each S_λ is compact, and hence Z is locally compact. Since every quotient space and open subspace of a k-space is a k-space, Y is a k-space. Therefore, by pursuing the proof of Theorem 1, we have Theorem 2.

To prove the implication (b) $\to$ (a) of Theorems 3 and 4, we need a theorem of Hušek [**4**, Theorem 3]. His theorem can be restated as follows:

HUŠEK'S THEOREM. *For a space X the following conditions are equivalent*:
(a) card X *is nonmeasurable.*
(b) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ *holds for each discrete space* Y.

PROOF OF THEOREM 3. (a) $\to$ (b) is the result of Comfort quoted in the introduction. (b) $\to$ (a). By Hušek's theorem, card X is nonmeasurable. It follows from Theorem 1 and [**6**, Theorem 5.2] that X is locally compact and realcompact.

PROOF OF THEOREM 4. (b) $\to$ (a). Since a discrete space is a k-space, by Husek's Theorem, card X is nonmeasurable. By Theorem 2, υX is locally pseudocompact, and hence is locally compact, because every pseudocompact realcompact space is compact (cf. [**3**, 8E1]).

3. Remarks. (1) If υX is locally compact, then X is locally pseudocompact, but the converse is false (see [**1**]).

(2) The space Y constructed in the proof of Theorems 1 and 2 and [**6**, Theorem 5.2] is 0-dimensional (i.e., ind $Y = 0$). Hence all theorems in this note remain true if "for each (k-) space Y" is replaced by "for each 0-dimensional (k-) space Y".

(3) A space similar to the space S_λ in the proof of Theorem 1 was used in **[6]** to show that every member of $\mathfrak{R}$ is realcompact.

(4) A space X is said to be *topologically complete* if it is complete with respect to its finest uniformity (i.e., $X = \mu X$). In **[7]**, Morita proved that if X is locally compact topologically complete, then $\mu(X \times Y) = \mu X \times \mu Y$ holds for each space Y, and Isiwata **[5]** proved that if $\mu(X \times Y) = \mu X \times \mu Y$ holds for each space Y, then X is topologically complete (cf. also **[8]**). Hence the analogous results of Theorems $1 \sim 4$ remain true, with no cardinality conditions, for topological completions (in this case, we need to use [**5**, Theorem 2.3], [**7**, Theorem 3.1] and [**2**, Lemma 3.1] instead of Theorem 4, (a) $\rightarrow$ (b), [**3**, 8A4] and [**3**, 8E1], respectively).

ADDED IN PROOF. Recently, Blair and Hager (*z-embedding in $\beta X \times \beta Y$, Set theoretic topology*, Academic Press, New York, 1977) asked whether the following condition (d′) implies that $X \times Y$ is z-embedded in $\beta X \times \beta Y$ (i.e., each zero-set of $X \times Y$ is the trace on $X \times Y$ of a zero-set of $\beta X \times \beta Y$):

(d′) For every real-valued continuous function f on $X \times Y$ and every $\varepsilon > 0$, there is a countable open rectangular cover $\{G_n\}$ of $X \times Y$ such that $\sup\{|f(p) - f(q)| \,|\, p, q \in G_n\} < \varepsilon$ for each n.

In the same paper, they proved that if X has a countable base, then $X \times Y$ satisfies (d′) for each space Y, and that if $X \times Y$ is z-embedded in $\beta X \times \beta Y$, then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds. From these facts, since there exists a nonlocally compact space with a countable base, Theorem 3 answers this question negatively. Furthermore, combining Theorem 3 with their results (3.2, 3.3), we obtain: X is a locally compact space with a countable base if and only if $X \times Y$ is z-embedded in $\beta X \times \beta Y$ for each space Y.

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