ON THE NUMBER OF LATTICE POINTS IN PLANAR DOMAINS

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ABSTRACT. Lattice-point estimates are obtained for rotations of planar sets.

Let C be a compact subset of the plane, such that its boundary ∂C is of class C^∞ and has finitely many points of zero curvature. Assume additionally that the order of contact at each point of zero curvature is finite and that C is star-like with respect to the origin in the sense that no normal line to ∂C is perpendicular to the radial vector. For a fixed $\theta \in [0, 2\pi]$, let L_θ be the image of the integral lattice points under a clockwise rotation of size θ . For $x > 0$, define $N(x, \theta)$ to be the number of points in L_θ which intersect xC . Now, let $R(x, \theta) = N(x, \theta) - Vx^2$, where $V = \int_C dx_1 dx_2$.

Suppose for some $\Delta > 0$, we draw bands of width $2 + \Delta$ symmetrically about each normal line to ∂C at points of zero curvature. Let B be the union of these bands.

THEOREM 1. *There exists $M > 0$ and a function $\Phi(\theta)$ which is L^1 on S^1 , such that $|R(x, \theta)| \leq Mx^{2/3} + \Phi(\theta)x^{1/2}$.*

In the next theorem we shall for simplicity assume that ∂C has only one point of zero curvature located on the positive x_1 -axis and such that the normal line is also on the x_1 -axis. These hypotheses can be generalized.

THEOREM 2. *Under the above conditions $|R(x, \theta)| = O(x^{2/3})$ for each $\theta \neq \pi/2, 3\pi/2$ having an irrational, algebraic tangent.*

REMARKS. (1) A result similar to Theorem 1 has recently been obtained by Y. Colin de Verdière [3].

(2) I would like to thank Professor Burton Randol for the initial suggestions, help and encouragement he gave me while I was working on this paper.

We shall proceed now to prove Theorem 1. If $X = (x_1, x_2)$ and $J(X)$ is the characteristic function of the set C , then $J(X/x)$ is the characteristic function of the set xC . Suppose $\hat{J}(Y)$ is the Fourier transform of $J(X)$, then $x^2 \hat{J}(xY)$ is the Fourier transform of $J(X/x)$. If we think of θ as a transformation in $SO(2)$, we can write $N(x, \theta) = \sum_N J(\theta(N)/x)$.

Next define $\delta(Y)$ to be a nonnegative C^∞ function with support in the unit

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disk, and satisfying $\int_R \delta(Y) dV_Y = 1$. Let $\delta_\epsilon(Y) = \epsilon^{-2} \delta(Y/\epsilon)$. Now $\delta_\epsilon(Y)$ has support in the disk $|Y| \leq \epsilon$, and its integral is also 1. If we define $J_\epsilon(x, Y) = \int_R \delta_\epsilon(Y - X) J(X/x) dV_X$ and set $N_\epsilon(x, \theta) = \sum J_\epsilon(x, \theta(N))$ we can apply the Poisson summation formula to find that

$$\sum J_\epsilon(x, \theta(N)) = \sum \hat{\delta}_\epsilon(\theta(N)) [x^2 \hat{J}(x\theta(N))],$$

since $J_\epsilon(x, Y)$ is a C^∞ function with compact support. I.e.,

$$N_\epsilon(x, \theta) = Vx^2 + \sum' \hat{\delta}_\epsilon(\theta(N)) [x^2 \hat{J}(x, \theta(N))]$$

where $\sum'$ indicates summation over all nonzero integral lattice points.

Assuming, as we clearly may, that the distance from the origin to ∂C is initially large enough, we can say that for $\epsilon > 0$, $N_\epsilon(x - \epsilon, \theta) \leq N(x, \theta) \leq N_\epsilon(x + \epsilon, \theta)$ by the star-like conditions on C . This implies that

$$N_\epsilon(x - \epsilon, \theta) - Vx^2 \leq R(x, \theta) \leq N_\epsilon(x + \epsilon, \theta) - Vx^2.$$

By the right-hand side,

$$R(x, \theta) \leq V[(x + \epsilon)^2 - x^2] + \sum' \hat{\delta}_\epsilon(\theta(N)) [(x + \epsilon)^2 \hat{J}((x + \epsilon)\theta(N))].$$

For a given N , let $\alpha_1, \dots, \alpha_k$ be the set of angles, with each $\alpha_j \in [0, \pi/2]$, which the line through the origin and $\theta(N)$ makes with the normal lines to ∂C at points of zero curvature. Set $[A(\theta(N))] = \min_{1 \leq j \leq k} \alpha_j$. By Theorem 1 of [1] our hypotheses imply that $\sup_x x^{3/2} |\hat{J}(x\theta(N))|$ is bounded by a fixed multiple of $[A(\theta(N))]^{-(n_0-1)/2n_0}$, where n_0 is the largest order of contact between ∂C and a tangent line.

Thus $x^2 \hat{J}(x\theta(N)) \leq x^{1/2} |N|^{-3/2} M_1 [A(\theta(N))]^{-h}$ where $h = (n_0 - 1)/2n_0$. Observe $h < \frac{1}{2}$.

Also, $|\hat{\delta}_\epsilon(Y)| \leq M_2(1 + \epsilon|Y|)^{-1}$ for $M_2 > 0$, so we can write

$$\begin{aligned} |R(x, \theta)| &\leq V[(x + \epsilon)^2 - x^2] \\ &\quad + M_3(x + \epsilon)^{1/2} \sum' (1 + \epsilon|N|)^{-1} |N|^{-3/2} [A(\theta(N))]^{-h}. \end{aligned}$$

Now, if we let $\epsilon = x^{-1/3}$, we obtain the following:

$$\begin{aligned} (a) \quad R(x, \theta) &\leq \left\{ O(x^{2/3}) + M_3(x + \epsilon)^{1/2} \right. \\ &\quad \cdot \sum' (1 + \epsilon|N|)^{-1} |N|^{-3/2} [A(\theta(N))]^{-h} \left. \right\} \end{aligned}$$

where "O" is independent of θ .

Similarly we can show that

$$\begin{aligned} (b) \quad R(x, \theta) &\geq - \left\{ O(x^{2/3}) + M_3(x - \epsilon)^{1/2} \right. \\ &\quad \cdot \sum' (1 + \epsilon|N|)^{-1} |N|^{-3/2} [A(\theta(N))]^{-h} \left. \right\}. \end{aligned}$$

(a) and (b) imply that

$$|R(x, \theta)| \leq M_4 \left\{ x^{2/3} + x^{1/2} \sum' (1 + \varepsilon|N|)^{-1} |N|^{-3/2} [A(\theta(N))]^{-h} \right\}$$

for some $M_4 > 0$.

Now write $\Sigma' = \Sigma^* + \Sigma^{**}$ where Σ^* indicates that the sum is taken over all N for which $\theta(N)$ is in the complement of B , and Σ^{**} means $\theta(N)$ is in B .

We shall first estimate $\Sigma^*(1 + \varepsilon|N|)^{-1} |N|^{-3/2} [A(\theta(N))]^{-h}$. Now $\Sigma^* = \Sigma_{|N| < 1/\varepsilon}^* + \Sigma_{|N| > 1/\varepsilon}^*$ where, as before, $\varepsilon = x^{-1/3}$. It can be easily shown that for all $\theta(N)$ in the complement of B , $f(r, \varphi) = (1 + \varepsilon r)^{-1} r^{-3/2} \alpha(\varphi)^{-h}$, evaluated at $\theta(N)$ is less than $M_5 \int_{\square} f(r, \varphi) r dr d\varphi$ for some $M_5 > 0$, where $\square$ is a square centered at $\theta(N)$, and $\alpha(\varphi)$ is the smallest angle in $[0, \pi/2]$ which the line through the origin of argument φ makes with the normal lines to ∂C at points of zero curvature. Thus the two sums $\Sigma_{|N| < 1/\varepsilon}^*$ and $\Sigma_{|N| > 1/\varepsilon}^*$ may be estimated by comparing them with integrals. Those estimates are the same and the joint estimate is $O(x^{1/6})$. So

$$x^{1/2} \Sigma^* (1 + \varepsilon|N|)^{-1} |N|^{-3/2} [A(\theta(N))]^{-h} \leq M_6 x^{2/3}.$$

Now we shall estimate

$$\Phi(\theta) = \Sigma^{**} (1 + \varepsilon|N|)^{-1} |N|^{-3/2} [A(\theta(N))]^{-h}.$$

(Recall Σ^{**} indicates that the sum is taken for all N such that $\theta(N)$ is in B .) It is sufficient to estimate this sum for $\theta(N)$ in one of the bands in B . In doing this we may without loss of generality assume that this band is centered about the x_1 -axis and that $0 \leq \theta \leq \pi/4$. Let $\mu = \tan(-\theta)$. Now $\Phi(\theta)$ is clearly bounded by a multiple of

$$\Sigma^{**} (1 + \varepsilon|N|)^{-1} |N|^{-3/2} |\tan \theta(N/|N|)|^{-h},$$

where we are identifying the point $\theta(N/|N|)$ on S^1 with an angle in the obvious way. The last sum is easily seen to be bounded by a multiple of

$$\Sigma^{***} (1 + \varepsilon|N|)^{-1} |N|^{-3/2} |\mu - \tan(N/|N|)|^{-h}$$

where the sum is taken over the nonzero lattice points in the band obtained by rotating the original band through $-\theta$. This last sum is dominated by a multiple of $\sum_{q=1}^{\infty} q^{-3/2} |\mu - P/q|^{-h}$, where, for each q , p is so selected that $|\mu - P/q|$ is minimal. Now the last sum can be written as $\sum_{q=1}^{\infty} q^{-3/2+h} |q\mu - p|^{-h} = \sum_{q=1}^{\infty} q^{-3/2+h} \|q\mu\|^{-h}$, where " $\| \cdot \|$ " denotes the distance to the nearest integer.

Consider $\int_0^1 \|q\mu\|^{-h} d\mu = I$. I is finite and independent of q .

Now $\sum_q q^{-3/2+h} \int_0^1 \|q\mu\|^{-h} d\mu$ is convergent since $h < 1/2$. Thus by Beppo-Levi's Theorem, $\sum q^{-3/2+h} \|q\mu\|^{-h}$ is convergent to an L^1 function for almost all $\mu \in [0, 1]$. This implies the desired result.

PROOF OF THEOREM 2. Suppose $\theta \neq \pi/2, 3\pi/2$ has an algebraic, not rational tangent. With notation as before, we only need to estimate

$$x^{1/2} \sum_q q^{-3/2} |1 + \varepsilon q|^{-1} |\mu - P/q|^{-h} \quad (\varepsilon = x^{-1/3}).$$

By Roth's Theorem [2], if μ is any algebraic number (not rational) and if $|\mu - P/q| < 1/q^\beta$ has infinitely many solutions, then $\beta < 2$. I.e., if $\beta > 2$, then $|\mu - P/q| < 1/q^\beta$ has finitely many solutions. In particular if $\eta > 0$ is such that $h \leq 1/(2 + \eta)$, $|\mu - P/q| \geq q^{-(2+\eta)}$ for all but a finite number of q 's. Thus

$$\sum_q q^{-3/2} |1 + \varepsilon q|^{-1} |\mu - P/q|^{-h} \\ \leq M_7 + \sum_q q^{-3/2} |1 + \varepsilon q|^{-1} q^{(2+\eta)h} = M_7 + \sum_{q < 1/\varepsilon} + \sum_{q > 1/\varepsilon}.$$

The two sums can be estimated by comparing them with integrals. The estimates are equal, and the joint estimate, when multiplied by $x^{1/2}$, is $O(x^{(2+\eta)h/3 + 1/3})$, which proves Theorem 2, since $(2 + \eta)h/3 + 1/3 \leq 2/3$.

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