A RESTRICTION THEOREM FOR SPACE CURVES

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ABSTRACT. Let α be a C^k curve ($k \geq 3$) in R^3 with nonvanishing curvature and torsion. It is proved that the restriction operator $T: f \rightarrow f|_\alpha$ is bounded from $L^p(R^3)$ to $L^q(\alpha)$ if $1 < p < 15/13$ and $1/q > 6(1 - 1/p)$, and that T is not bounded if $p \geq 6/5$ or $1/q < 6(1 - 1/p)$.

Let α be a C^k curve in R^3 ($k \geq 3$). For $f \in C_0^\infty(R^3)$ define the restriction operator T by

$$Tf = \hat{f}|_\alpha.$$

Then the following result holds:

THEOREM. *If the curve α has nonvanishing curvature and torsion, the inequality*

$$(1) \quad \|Tf\|_{L^q(\alpha)} \leq C_p \|f\|_{L^p(R^3)}$$

holds if $1 < p < 15/13$ and $1/q > 6(1 - 1/p)$. The inequality does not hold if $p \geq 6/5$ or $1/q < 6(1 - 1/p)$.

PROOF. First we show that (1) does not hold if $1/q < 6(1 - 1/p)$. Let f be a smooth function such that its Fourier transform $\hat{f}$ is supported on a parallelepiped centered at any point P on the curve and whose dimensions are η, η^2, η^3 along the tangent, normal and binormal direction at P , respectively (η denotes a small number). Then up to infinitesimals of higher order, $\|f\|_{L^p(R^3)} = \eta^{6(1-1/p)}$ and $\|Tf\|_{L^q(\alpha)} = \eta^{1/q}$, hence T cannot be a bounded operator unless $1/q \geq 6(1 - 1/p)$.

Now we prove (1). We may assume, without loss of generality, that α is defined by the equations $(t, \varphi(t), \psi(t), 0 < t < \eta)$, where η is a small number and $\varphi(t) = kt^2/2 + \xi(t)$, $\psi(t) = k\tau t^3/6 + \zeta(t)$. Here k and τ denote curvature and torsion, $\xi(t)$ and $\zeta(t)$ are infinitesimals of third and fourth order w.r.t. t . Then denote by T^* the adjoint operator

$$(T^*f)^3(y) = \int_0^\eta \int_0^\eta \int_0^\eta \exp \left(-i \left(y_1 \left(\sum_i t_i \right) + y_2 \left(\sum_i \varphi(t_i) \right) + y_3 \left(\sum_i \psi(t_i) \right) \right) \right) \cdot f(t_1) f(t_2) f(t_3) dt_1 dt_2 dt_3$$

with $y = (y_1, y_2, y_3)$.

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If we make the change of variables $u_1 = \sum_i t_i$, $u_2 = \sum_i \varphi(t_i)$, $u_3 = \sum_i \psi(t_i)$, we obtain

$$(T^*f)^3(y) = \int_D \exp(-i(y_1 u_1 + y_2 u_2 + y_3 u_3)) f(t_1) f(t_2) f(t_3) \cdot |J(t_1, t_2, t_3)|^{-1} du_1 du_2 du_3,$$

where D denotes the image of the cube of side η in the (u_1, u_2, u_3) -plane and J the Jacobian of the transformation. Now we apply the Hausdorff-Young inequality and we change variables once more. In so doing we get

$$\begin{aligned} \|T^*f\|_{L^{p^*}(R^3)}^3 &= \|(T^*f)^3\|_{L^{p^*}(R^3)} \\ &\leq C_p \left(\int_D |f(t_1) f(t_2) f(t_3)| |J(t_1, t_2, t_3)|^{-1} \right)^{p^*/(p^*-3)} du_1 du_2 du_3 \\ &= C_p \left(\int_0^\eta \int_0^\eta \int_0^\eta |f(t_1) f(t_2) f(t_3)|^{p^*/(p^*-3)} \right. \\ &\quad \left. \cdot |J(t_1, t_2, t_3)|^{-p^*/(p^*-3)+1} dt_1 dt_2 dt_3 \right)^{(p^*-3)/p^*} \end{aligned}$$

By the Vandermonde determinant

$$\begin{aligned} |J(t_1, t_2, t_3)| &= \frac{k^2 \tau}{12} \begin{vmatrix} 1 & 1 & 1 \\ t_1 + \xi'(t_1) & t_2 + \xi'(t_2) & t_3 + \xi'(t_3) \\ t_1^2 + \xi''(t_1) & t_2^2 + \xi''(t_2) & t_3^2 + \xi''(t_3) \end{vmatrix} \\ &> c |(t_1 - t_2)(t_1 - t_3)(t_2 - t_3)|. \end{aligned}$$

Now set $a = 3/(p^* - 3)$ and $F = |f|^{p^*/(p^*-3)}$. By the change of variables $v_1 = t_1$, $v_2 = t_1 - t_2$, $v_3 = t_1 - t_3$, and by Hölder's inequality, we obtain

$$\begin{aligned} &\int \int \int F(v_1) F(v_1 - v_2) F(v_1 - v_3) |v_2 v_3 (v_2 - v_3)|^{-a} dv_1 dv_2 dv_3 \\ &\leq \|F\|_{L^3(R^3)}^3 \int \int |v_2 v_3 (v_2 - v_3)|^{-a} dv_2 dv_3 \leq C_1 \|F\|_{L^3(R^3)}^3, \end{aligned}$$

when $a < 2/3$. Hence

$$\|T^*f\|_{L^{p^*}(R^3)}^3 \leq C_p \|f\|_{L^{3p^*/(p^*-3)}(\alpha)}^3$$

for $p^* > 15/2$. By duality this proves, in particular, the following inequality:

$$\|Tf\|_{L^{5/4+\epsilon}(\alpha)} \leq C \|f\|_{L^{15/13-\epsilon}(R^3)}$$

(ξ' and ξ'' denote small positive numbers). Now (1) follows by interpolating this estimate with the trivial one $\|Tf\|_{L^\infty(\alpha)} \leq C_2 \|f\|_{L^1(R^3)}$.

Finally, if in (2) we take $p^* = 6$ and f to be the constant 1, using Plancherel's theorem we see that T^*f does not belong to $L^6(R^3)$; hence T is not a bounded operator from $L^p(R^3)$ to $L^q(\alpha)$ for any q if $p > 6/5$. Note that the theorem does not take care of the case $1/q > 6(1 - 1/p)$ and $15/13 \leq p < 6/5$, or the case $1/q = 6(1 - 1/p)$ and $1 \leq p \leq 15/13$. The

reader should compare our techniques with those of [1]–[4].

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