STONE'S THEOREM AND SPECTRAL SUBSPACES OF AUTOMORPHISMS

ERLING STØRMER

ABSTRACT. It is shown that spectral subspaces of automorphisms of a von Neumann algebra can be defined by use of Stone's theorem on unitary representations.

It is well known that the theory of spectral subspaces of automorphisms as developed by Arveson [1] generalizes Stone's theorem for unitary representations. In this note we shall show a converse result, thus indicating how the theory of spectral subspaces of abelian groups of automorphisms of von Neumann algebras, see [1], [2], [3], can be developed from Stone's theorem. The idea is that a $*$ -automorphism of the bounded operators $B(H)$ on a Hilbert space H is implemented by a unitary operator, and so restricts to an isometry of the Hilbert-Schmidt operators H_2 on H . Thus Stone's theorem can be used on unitary representations on H_2 and then "lifted" to $B(H)$.

Throughout this note we let G be a locally compact abelian group with Haar measure dt and dual group Γ ; $t \rightarrow u_t$ is a strongly continuous unitary representation of G on the Hilbert space H , and $\alpha_t = \text{Ad}(u_t)$. Then $t \rightarrow \alpha_t$ is a continuous representation of G into the automorphism group $\text{Aut } B(H)$ of $B(H)$, i.e., $t \rightarrow \varphi(\alpha_t(x))$ is a continuous function for all $x \in B(H)$, $\varphi \in B(H)_*$. Recall from [1], [2], [3] that if $f \in L^1(G)$ and $x \in B(H)$ then

$$\pi_\alpha(f)(x) = \int_G f(t) \alpha_t(x) dt, \quad Z(f) = \{\gamma \in \Gamma: \hat{f}(\gamma) = 0\},$$

$$\text{Sp}_\alpha(x) = \bigcap \{Z(f): f \in L^1(G), \pi_\alpha(f)(x) = 0\}.$$

We assume M is a von Neumann algebra acting on H such that $\alpha_t(M) = M$, $t \in G$. Then if E is a closed subset of Γ , its spectral subspace is

$$M^\alpha(E) = \{x \in M: \text{Sp}_\alpha(x) \subset E\}.$$

Moreover, the subspaces $M^\alpha(E)$ determine α [1].

LEMMA. Denote by $\tilde{\alpha}_t$ the restriction of α_t to the Hilbert-Schmidt operators H_2 on H . Then $t \rightarrow \tilde{\alpha}_t$ is a weakly, hence strongly, continuous unitary representation of G on H_2 .

PROOF. Let $x, y \in H_2$ and $\varepsilon > 0$. Let $y = y_1 + y_2$ with y_1 of finite rank and

Received by the editors January 12, 1977.

AMS (MOS) subject classifications (1970). Primary 46L10; Secondary 46L05, 43A25.

© American Mathematical Society 1978

$\|y_2\|_2 < \varepsilon/\|x\|_2$, where $\|z\|_2 = \langle z, z \rangle^{1/2}$, and $\langle, \rangle$ is the inner product on H_2 . Since $\alpha_t(x) \rightarrow x$ ultraweakly as $t \rightarrow e$, the identity in G , there is a neighborhood N of e in G such that $|\langle \tilde{\alpha}_t(x) - x, y_1 \rangle| < \varepsilon$ for $t \in N$. Then

$$\begin{aligned} |\langle \tilde{\alpha}_t(x) - x, y \rangle| &\leq |\langle \tilde{\alpha}_t(x) - x, y_1 \rangle| + |\langle \tilde{\alpha}_t(x) - x, y_2 \rangle| \\ &< \varepsilon + 2\|x\|_2\|y_2\|_2 < 3\varepsilon, \end{aligned}$$

proving the lemma.

By Stone's theorem applied to the continuous unitary representation $t \rightarrow \tilde{\alpha}_t$ on H_2 , there exists a projection valued measure P_λ on Γ with values in $B(H_2)$ such that $\tilde{\alpha}_t = \int_\Gamma \langle \lambda, t \rangle dP_\lambda$. If E and F are closed subsets of Γ we denote by $E + F$ the closure of the set $\{\gamma + \lambda : \gamma \in E, \lambda \in F\}$. We denote by $P(F)$ the closed subspace of H_2 obtained as the range of $\int_F dP_\lambda$, and by $P(F)^-$ its ultraweak closure in $B(H)$.

THEOREM. *With the above assumptions and notation, if E is a closed subset of Γ then $M^\alpha(E) = \bigcap_N \{M \cap P(E + N)^-\}$, where the intersection is taken over all compact neighborhoods N of the identity in Γ .*

PROOF. Let F be a closed subset of Γ . Let $x \in M \cap P(F)^-$ and (x_β) be a net in $P(F)$ which converges ultraweakly to x . Let $f \in L^1(G)$ have Fourier transform vanishing in a neighborhood of F . By [1, Proposition 1.6] $\pi_\alpha(f)$ is an ultraweakly continuous linear map on $B(H)$. Thus by [1, Remark, §2] $0 = \pi_{\tilde{\alpha}}(f)(x_\beta) \rightarrow \pi_\alpha(f)(x)$, where we have identified the operator $\pi_{\tilde{\alpha}}(f)$ defined by $\tilde{\alpha}$ on H_2 and its extension $\pi_\alpha(f)$ to $B(H)$. Thus $\pi_\alpha(f)(x) = 0$ for all such f , so again by [1], $x \in M^\alpha(F)$. Thus $\bigcap_N \{M \cap P(E + N)^-\} \subset \bigcap_N M^\alpha(E + N) = M^\alpha(E)$ [1, Proposition 2.2].

Conversely, let $x \in M^\alpha(E)$, so, in particular, $x \in B(H)^\alpha(E)$. If F is a closed subset of Γ let $R_2^\alpha(F)$ (resp. $R^\alpha(F)$) denote the closed (resp. ultraweakly closed) subspace of H_2 (resp. $B(H)$) generated by range $\pi_\alpha(f)$ in H_2 (resp. in $B(H)$) for all $f \in L^1(G)$ with $\text{supp } \hat{f}$ compact and contained in F . By [1, Proposition 2.2], $B(H)^\alpha(E) = \bigcap_N R^\alpha(E + N)$, where the intersection is taken over all compact neighborhoods N of the identity in Γ . Let $x \in R^\alpha(E + N)$ and assume there are $f \in L^1(G)$ such that $\text{supp } \hat{f}$ is compact and contained in $E + N$, and $y \in B(H)$, such that $x = \pi_\alpha(f)(y)$. Since H_2 is ultraweakly dense in $B(H)$ there is a net (y_β) in H_2 which converges ultraweakly to y . Since $\pi_\alpha(f)$ is ultraweakly continuous,

$$x = \pi_\alpha(f)(y) = \lim_{\beta} \pi_\alpha(f)(y_\beta) \in R_2^\alpha(E + N)$$

But from the theory of spectral subspaces applied to unitary representations [1], $R_2^\alpha(F) = P(F)$ for all closed sets $F \subset \Gamma$. Thus $x \in P(E + N)^-$ for all N , and since such x are dense in $R^\alpha(E + N)$, the proof is complete.

REMARK 1. We cannot sharpen the theorem to a statement like " $M^\alpha(E) = M \cap P(E)^-$ ". Indeed, let H be a separable Hilbert space and u a unitary operator on H such that the von Neumann algebra $\mathcal{A}$ generated by u is a maximal abelian subalgebra of $B(H)$ without minimal projections. Let α be

the representation of the integers defined by $\alpha_n = \text{Ad}(u^n) \in \text{Aut } B(H)$. Then there is no nonzero $x \in H_2$ such that $\alpha_n(x) = x$ for all n , so $P(\{1\}) = \{0\}$, while $B(H)^\alpha(\{1\}) = A$.

REMARK 2. The main idea in the proof of the theorem was to consider the restriction $\tilde{\varphi}$ of a map $\varphi \in B(B(H))$, the bounded linear maps of $B(H)$ into itself, to H_2 . If $\tilde{\varphi}$ is a bounded normal linear operator on H_2 we can do spectral theory for $\tilde{\varphi}$ in $B(H_2)$. It is tempting to generalize the above theorem and try to "lift" spectral theory for $\tilde{\varphi}$ in $B(H_2)$ to that of φ in $B(B(H))$. This, however, seems to be quite hopeless except in special cases. Indeed, while the norm $\|\varphi\|$ of φ in $B(B(H))$ is never smaller than the norm $\|\tilde{\varphi}\|$ of $\tilde{\varphi}$ in $B(H_2)$, there is no finite constant $k > 0$ such that $\|\varphi\| \leq k\|\tilde{\varphi}\|$ for all such φ .

REFERENCES

1. W. Arveson, *On groups of automorphisms of operator algebras*, J. Functional Analysis **15** (1974), 217–243.
2. H. J. Borchers, *Über Ableitungen von C^* -algebren*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II **2** (1973).
3. A. Connes, *Une classification des facteurs de type III*, Ann. Sci. École Norm. Sup. **6** (1973), 133–252.

MATEMATISK INSTITUTT POSTBOKS 1053 BLINDERN-OSLO 3, NORWAY