REMARK ON THE ZERO LOCUS OF A SMOOTH FUNCTION

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ABSTRACT. A sufficient condition is given to insure the existence of some smooth points in the zero set of a C^∞ function.

Since any closed set in $\mathbf{R}^n$ is the zero locus of a C^∞ function, there is little to be said, in general, about such a set. Even if the function f vanishes to finite order at the origin, i.e., $D^\alpha f(0) \neq 0$ for some α , the zero set Z of f near 0 still need not have much geometric structure.

EXAMPLE. Let $Y \subset \mathbf{R}$ be a Cantor set, and let $\varphi(y)$ be a C^∞ function on $\mathbf{R}$, nonnegative, zero only along Y . $f(x, y) = x^2 + \varphi(y)$ is zero only along $\{(0, y) | y \in Y\}$.

However, if the zero locus of an f vanishing to finite order is in some sense a boundary between two regions, we can say more.

THEOREM. Let $f \in C^\infty$, $f(0) = 0$; $D^\alpha f(0) \neq 0$, some α . Suppose $0 \in \bar{\Omega}^+ \cap \bar{\Omega}^-$, where $\Omega^\pm = \{x | \pm f(x) > 0\}$. Then there is an open subset Θ of Z where, for each $p \in \Theta$, there is a neighborhood U of p in $\mathbf{R}^n$ such that $U \cap Z$ is a C^∞ submanifold of $\mathbf{R}^n$, and $0 \in \Theta$.

PROOF. Malgrange's preparation theorem says there exist coordinates $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n)$ at 0 such that

$$f(x) = h(x)(x_n^k + a_{k-1}(x')x_n^{k-1} + \dots + a_0(x')) = h(x)P_{x'}(x_n)$$

with h and the $a_i \in C^\infty$, $h(0) \neq 0$. Let π denote the projection from $\mathbf{R}^n \rightarrow \mathbf{R}^{n-1}$, $(x', x_n) \rightarrow x'$, and set $\omega^\pm = \pi(\Omega^\pm)$; $\omega^\pm$ are open in $\mathbf{R}^{n-1}$. Take any neighborhood $V \times (-\varepsilon, \varepsilon)$ about 0, where $\varepsilon > 0$ and V is connected and open in $\mathbf{R}^{n-1}$. We first note that $\omega^+ \cap \omega^- \neq \emptyset$ in V : otherwise, $\exists x'$ such that $\pi^{-1}(x') \cap (\Omega^+ \cup \Omega^-) = \emptyset$, i.e., $P_{x'}(x_n)$ is identically zero, a contradiction. Now $P_{x'}(x_n)$ has zeroes over every point of $\omega^+ \cap \omega^-$, and we have only to isolate branches of these zeroes.

For $x' \in \omega^+ \cap \omega^-$, let $N(x') =$ the number of odd zeroes of $P_{x'}(x_n) =$ the number of roots ξ of $P_{x'}(x_n) = 0$ with odd multiplicity. $N(x')$ is integer-valued, lower-semicontinuous, positive and bounded by k on $\omega^+ \cap \omega^-$. Let $\omega \subset \omega^+ \cap \omega^-$ be the open subset where $N(x')$ assumes its maximum on $\omega^+ \cap \omega^-$. Next, consider for $x' \in \omega^+ \cap \omega^-$,

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$$M(x') = \sum \text{multiplicity}(\xi),$$

the summation taken over all odd roots ξ of $P_{x'}(x_n) = 0$. It is easy to see that $M(x')$ is integer-valued, upper-semicontinuous, positive and bounded on ω . Let ω' be the open subset of ω where $M(x')$ assumes its minimum value on ω .

For $x' \in \omega'$, let ξ' be an odd zero of $P_{x'}(x_n)$. Then, given $\varepsilon > 0$, if $|x'' - x'| < \delta$, $x'' \in \mathbb{R}^{n-1}$ and δ small enough, $N(x') = N(x'')$ and there is a *unique* odd zero ξ'' of $P_{x''}(x_n)$ with $|\xi' - \xi''| < \varepsilon$, and we have $\text{multiplicity}(\xi'') < \text{multiplicity}(\xi')$. If δ is small enough, however, $M(x'') = M(x')$, and

$$d = \text{multiplicity}(\xi'') = \text{multiplicity}(\xi')$$

and ξ'' is the unique zero of $P_{x''}(x_n)$ within ε of ξ' . Thus, in the cylinder

$$U = \{(y', y_n) | |y' - x'| < \delta, |x_n - y_n| < \varepsilon\},$$

the set

$$Z = \{(x', x_n) | f(x', x_n) = 0\} = \{(x', x_n) | P_{x'}(x_n) = 0\}$$

is identical with the set $\{(x', x_n) | (\partial^{d-1}/\partial x_n^{d-1})P_{x'}(x_n) = 0\}$, which is a smooth submanifold of U .

It seems unlikely that a more precise statement than the above can be made, e.g., about the measure-theoretic size of the set Θ . Its Hausdorff 1-dimensional density at 0, for example, can be made arbitrarily small. It is easy to see that the hypothesis $D^a f(0) \neq 0$ cannot be dropped from the statement of the theorem.

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