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ZONES OF UNIFORM DECOMPOSITION IN TENSOR PRODUCTS[1]

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ABSTRACT. Let V_λ be a finite dimensional irreducible module for a complex semisimple Lie algebra. It is shown that the decomposition of tensor products $V_\lambda \otimes V_\tau$ for all dominant integral weights τ may be derived from those for a finite set of such τ. An explicit choice of such a finite set (depending on λ) is given.

Introduction. Let L be a complex semisimple Lie algebra with simple roots $\{\alpha_1, \ldots, \alpha_l\}$ and fundamental weights $\{\omega_1, \ldots, \omega_l\}$. That is, $\{\omega_1, \ldots, \omega_l\}$ is a basis of the integral weight lattice, Λ, such that $\langle \omega_i, \alpha_j \rangle = 2(\omega_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$. By definition, $\tau = \sum_{i=1}^{l} m_i \omega_i \in \Lambda^+$ if and only if $m_i \geqslant 0$ are all integers. Also, $\sum_{i=1}^{l} \omega_i = \delta = \frac{1}{2} \sum \alpha$, where $\alpha \in \Phi^+$ is the set of all positive roots. All L-modules in this paper are finite dimensional.

Let W denote the Weyl group of L. W is generated by the simple reflections $\{\sigma_1, \ldots, \sigma_l\}$, where $\sigma_i(x) = x - \langle x, \alpha_i \rangle \alpha_i$. For any i, $1 \leqslant i \leqslant l$, we define $W(i)$ to be the subgroup of W generated by $\{\sigma_j | j \neq i, 1 \leqslant j \leqslant l\}$. Note that each element of $W(i)$ fixes ω_i.

In all of what follows, the set of weights of the irreducible L-module V_λ will be denoted by Π.

We shall prove

THEOREM 1. *Let V_λ be the irreducible L-module of highest weight λ. Let $\tau = \sum_{j=1}^{l} m_j \omega_j \in \Lambda^+$ and $V_\lambda \otimes V_\tau = \sum_{\gamma \in \Lambda^+} r_\gamma V_\gamma$. Then for each i, $1 \leqslant i \leqslant l$, there is a positive integer n_i, depending only on λ, such that if $m_i \geqslant n_i$, then $V_\lambda \otimes V_{\tau + \omega_i} = \sum_{\gamma \in \Lambda^+} r_\gamma V_{\gamma + \omega_i}$.*

We shall give explicit values for the n_i in terms of λ.

Theorem 1 should be compared with a result of Kostant [4]. He puts a much stronger requirement on τ, namely that $\mu + \tau$ is dominant for every $\mu \in \Pi$. Under this condition, one can read off the decomposition of $V_\lambda \otimes V_\tau$ from the weight-space decomposition of V_λ: $V_\lambda \otimes V_\tau = \sum_{\mu \in \Pi} \mathrm{Mult}_\lambda(\mu) V_{\mu + \tau}$. The conclusion of Theorem 1 clearly follows for such τ. However, Kostant's condition is satisfied only by dominant weights τ well into the interior of the fundamental chamber, and gives no information about infinitely many weights on or near the chamber walls. Theorem 1, on the other hand,

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expresses a condition of uniformity along lines in the decomposition of the tensor product $V_\lambda \otimes V_\tau$ whenever τ is outside a specified finite region.

If we let $S(i) = \bigcup_{\sigma \in W(i)} \sigma(\Lambda^+)$, then n_i may be chosen as the least positive integer such that for each $\mu \in \Pi$ we have $\mu + n_i\omega_i \in S(i)$.

COROLLARY 1. *Let V_λ be fixed. Let $(n_1, \ldots, n_l)$ be the l-tuple of positive integers which can be found by the above theorem. If we know the decompositions into irreducible L-modules of the finite set of tensor products $\{V_\lambda \otimes V_\tau | \tau = \sum_{j=1}^{l} m_j\omega_j$ and $m_j \leqslant n_j$ for all j, $1 \leqslant j \leqslant l\}$, then we know the decomposition of the tensor product of V_λ with any irreducible L-module.*

Let i, $1 \leqslant i \leqslant l$, be fixed throughout the following and let $S = S(i)$.

LEMMA 1. $\bigcup_{\sigma \in W(i)} \sigma(\Lambda^+) = \{x \in \Lambda | (x, \sigma\alpha_i) \geqslant 0, \forall \sigma \in W(i)\}$.

PROOF. Let $S = \bigcup_{\sigma \in W(i)} \sigma(\Lambda^+)$ and $S' = \{x \in \Lambda | (x, \sigma\alpha_i) \geqslant 0, \ \forall \sigma \in W(i)\}$. If $x \in \Lambda^+$ then $(x, \alpha_j) \geqslant 0$ for $1 \leqslant j \leqslant l$, so for any $\alpha \in \Phi^+$, $(x, \alpha) \geqslant 0$. For any $\sigma \in W(i)$, $\sigma\alpha_i \in \Phi^+$ because it is certainly a root and has $+1$ as its α_i coefficient, so all coefficients are nonnegative. It follows that $(x, \sigma\alpha_i) \geqslant 0$; that is, $x \in S'$. Thus, $\Lambda^+ \subseteq S'$. For any $x \in S'$ and any σ, $\sigma' \in W(i)$, $(\sigma x, \sigma'\alpha_i) = (x, \sigma^{-1}\sigma'\alpha_i) \geqslant 0$ since $\sigma^{-1}\sigma' \in W(i)$. This means that if $x \in S'$ then $\sigma x \in S'$ for any $\sigma \in W(i)$. From $\Lambda^+ \subseteq S'$ we then get $S \subseteq S'$.

Suppose there is an $x \in S'$, $x \notin S$. In the finite set $\{\sigma x | \sigma \in W(i)\}$ let σx be chosen such that $(\sigma x, \delta)$ is maximal. Since $x \notin S$, $\sigma x \notin \Lambda^+$ and there is a j, $1 \leqslant j \leqslant l$, such that $(\sigma x, \alpha_j) < 0$. If $j \neq i$ then $\sigma_j \in W(i)$ and $\sigma_j\sigma \in W(i)$. But $(\sigma_j\sigma x, \delta) = (\sigma x, \sigma_j\delta) = (\sigma x, \delta - \alpha_j) = (\sigma x, \delta) - (\sigma x, \alpha_j) > (\sigma x, \delta)$, contradicting the choice of σx. So $j = i$ and $(x, \sigma^{-1}\alpha_i) = (\sigma x, \alpha_i) < 0$. But since $\sigma^{-1} \in W(i)$, this contradicts $x \in S'$, giving $S = S'$.

LEMMA 2. *There is an integer $n_i \geqslant 0$ such that for any $\mu \in \Pi$, $\mu + n_i\omega_i \in S$. The least such n_i is* $\mathrm{Max}\{\langle \mu, \alpha_i \rangle | \mu \in \Pi\}$.

PROOF. For any $\sigma \in W(i)$, $(n_i\omega_i, \sigma\alpha_i) = n_i(\sigma^{-1}\omega_i, \alpha_i) = n_i(\omega_i, \alpha_i) = n_i(\alpha_i, \alpha_i)/2$. The conditions on n_i equivalent to $\mu + n_i\omega_i \in S$ for all $\mu \in \Pi$ are $0 \leqslant (\mu + n_i\omega_i, \sigma\alpha_i) = (\mu, \sigma\alpha_i) + (n_i\omega_i, \sigma\alpha_i) = (\mu, \sigma\alpha_i) + n_i(\alpha_i, \alpha_i)/2$ for all $\mu \in \Pi$ and all $\sigma \in W(i)$. That is, $n_i \geqslant -2(\mu, \sigma\alpha_i)/(\alpha_i, \alpha_i) = -\langle \mu, \sigma\alpha_i \rangle = -\langle \sigma^{-1}\mu, \alpha_i \rangle = \langle \sigma^{-1}\mu, \sigma_i\alpha_i \rangle = \langle \sigma_i\sigma^{-1}\mu, \alpha_i \rangle$. Since Π is invariant under W, $\{\sigma_i\sigma^{-1}\mu | \mu \in \Pi, \ \sigma \in W(i)\} = \Pi$. We now have the finite number of conditions $n_i \geqslant \langle \mu, \alpha_i \rangle$ for all $\mu \in \Pi$ which has least solution

$$n_i = \mathrm{Max}\{\langle \mu, \alpha_i \rangle | \mu \in \Pi\} \geqslant 0.$$

LEMMA 3. *For any $\gamma_1, \gamma_2 \in S$, $\gamma_1 + \gamma_2 \in S$.*

PROOF. Clear from Lemma 1.

PROOF OF THEOREM 1. If we use the notation

$$T_\lambda = \sum_{\sigma \in W} \mathrm{sgn}(\sigma) \exp(\sigma(\lambda + \delta)),$$

then the Weyl character formula says $X_\lambda \cdot T_0 = T_\lambda$, where X_λ is the character of a representation V_λ of highest weight λ. Then the character of $V_\lambda \otimes V_\tau$ is $X_\lambda \cdot X_\tau$. After some elementary manipulations, one sees that

$$X_\lambda \cdot X_\tau \cdot T_0 = \sum_{\mu \in \Pi} \text{Mult}_\lambda(\mu) \cdot T_{\mu+\tau}.$$

Replacing τ by $\tau + \omega_i$, we also have

$$X_\lambda \cdot X_{\tau+\omega_i} \cdot T_0 = \sum_{\mu \in \Pi} \text{Mult}_\lambda(\mu) \cdot T_{\mu+\tau+\omega_i}.$$

By Lemma 2, $\mu + n_i\omega_i \in S$ for all $\mu \in \Pi$. If $\tau = \sum_{j=1}^{l} m_j\omega_j$ satisfies $m_i \geqslant n_i$, then $\tau - n_i\omega_i \in \Lambda^+ \subseteq S$. By Lemma 3, $\mu + \tau = (\mu + n_i\omega_i) + (\tau - n_i\omega_i) \in S$. Of course, ω_i, $\delta \in \Lambda^+ \subseteq S$ and so $\mu + \delta + \tau \in S$ as well as $\mu + \delta + \tau + \omega_i \in S$. This means that both $\mu + \delta + \tau$ and $\mu + \delta + \tau + \omega_i$ are conjugate by elements of $W(i)$ to dominant weights. In fact, they are conjugate by the same element because if $\sigma_\mu(\mu + \delta + \tau) \in \Lambda^+$ for $\sigma_\mu \in W(i)$ then $\sigma_\mu(\mu + \delta + \tau + \omega_i) = \sigma_\mu(\mu + \delta + \tau) + \omega_i \in \Lambda^+$. Thus

$$\begin{aligned} T_{\mu+\tau} &= \sum_{\sigma \in W} \text{sgn}(\sigma) \exp(\sigma(\mu + \tau + \delta)) \\ &= \sum_{\sigma \in W} \text{sgn}(\sigma)\, \text{sgn}(\sigma_\mu) \exp\big(\sigma(\sigma_\mu(\mu + \tau + \delta))\big) \end{aligned}$$

and

$$\begin{aligned} T_{\mu+\tau+\omega_i} &= \sum_{\sigma \in W} \text{sgn}(\sigma) \exp(\sigma(\mu + \tau + \omega_i + \delta)) \\ &= \sum_{\sigma \in W} \text{sgn}(\sigma)\, \text{sgn}(\sigma_\mu) \exp\big(\sigma(\sigma_\mu(\mu + \tau + \delta) + \omega_i)\big). \end{aligned}$$

This means that $T_{\mu+\tau}/T_0 = \text{sgn}(\sigma_\mu) \cdot X_{\sigma_\mu(\mu+\tau+\delta)-\delta}$ and $T_{\mu+\tau+\omega_i}/T_0 = \text{sgn}(\sigma_\mu) \cdot X_{\sigma_\mu(\mu+\delta+\tau)-\delta+\omega_i}$. Then

$$X_\lambda \cdot X_\tau = \sum_{\mu \in \Pi} \text{Mult}_\lambda(\mu) \cdot \text{sgn}(\sigma_\mu) \cdot X_{\sigma_\mu(\mu+\tau+\delta)-\delta}$$

and

$$X_\lambda \cdot X_{\tau+\omega_i} = \sum_{\mu \in \Pi} \text{Mult}_\lambda(\mu) \cdot \text{sgn}(\sigma_\mu) \cdot X_{\sigma_\mu(\mu+\tau+\delta)-\delta+\omega_i}.$$

Grouping equivalent terms together, if $X_\lambda \cdot X_\tau = \sum_{\gamma \in \Lambda^+} r_\gamma X_\gamma$ then the above shows that $X_\lambda \cdot X_{\tau+\omega_i} = \sum_{\gamma \in \Lambda^+} r_\gamma X_{\gamma+\omega_i}$ as claimed by the theorem.

It should be noted that the author originally based his proof on a formula of Klimyk [3], whose geometric nature was essential to the discovery of this result. The present proof, based on the closely related and well-known Weyl character formula, was suggested by the referee.

Note that if $\tau = \sum_{j=1}^{l} m_j\omega_j$ satisfies $m_j \geqslant n_j$ for all j, $1 \leqslant j \leqslant l$, then by Lemma 2, for each $\mu \in \Pi$ and each j, $\mu + n_j\omega_j \in S(j)$ and $\tau - n_j\omega_j \in \Lambda^+ \subseteq S(j)$. Then $\mu + \tau \in S(j)$, which means $\mu + \tau \in \bigcap_{1 \leqslant j \leqslant l} S(j)$ for all $\mu \in \Pi$.

For each j, $\Lambda^+ \subseteq S(j)$, so $\Lambda^+ \subseteq \bigcap_{1 \leqslant j \leqslant l} S(j)$. For each j, $S(j) \subseteq \{x \in$

$\Lambda|(x, \alpha_j) > 0\}$. This means that $\bigcap_{1 \leqslant j \leqslant l} S(j) \subseteq \{x \in \Lambda|(x, \alpha_j) \geqslant 0$ for all $1 \leqslant j \leqslant l\} = \Lambda^+$. We now have $\bigcap_{1 \leqslant j \leqslant l} S(j) = \Lambda^+$.

The above says that for each $\mu \in \Pi$, $\mu + \tau \in \Lambda^+$, which is the condition required by Kostant's theorem. Although Kostant used a theorem of Brauer **[1]** in his proof, Weyl's formula also gives the result as follows. If $\mu + \tau \in \Lambda^+$ then $\mu + \delta + \tau$ is strictly dominant, so $\sigma_\mu = 1$ and $X_{\sigma_\mu(\mu+\delta+\tau)-\delta} = X_{\mu+\tau}$. Weyl's formula then says $X_\lambda \cdot X_\tau = \Sigma_{\mu \in \Pi} \operatorname{Mult}_\lambda(\mu) \cdot X_{\mu+\tau}$ which is now a direct sum.

In fact, if $\tau_0 = \Sigma_{j=1}^{l} n_j\omega_j$ then no "smaller" dominant weight τ' satisfies $\mu + \tau' \in \Lambda^+$ for all $\mu \in \Pi$. If we let $\tau' = \Sigma_{j=1}^{l} q_j\omega_j$ with some $q_p < n_p$, then the condition $\mu + \tau' \in \Lambda^+$ for all $\mu \in \Pi$ means $\langle \mu + \tau', \alpha_i\rangle \geqslant 0$ for $1 \leqslant i \leqslant l$. That is, $\langle \mu, \alpha_i\rangle + \Sigma_{j=1}^{l} q_j\langle \omega_j, \alpha_i\rangle = \langle \mu, \alpha_i\rangle + q_i \geqslant 0$. Or $q_i \geqslant -\langle \mu, \alpha_i\rangle = \langle \mu, \sigma_i\alpha_i\rangle = \langle \sigma_i\mu, \alpha_i\rangle$ for all $\mu \in \Pi$ and $1 \leqslant i \leqslant l$. As in Lemma 2, $\{\sigma_i\mu|\, \mu \in \Pi\} = \Pi$, so $q_i \geqslant \langle \mu, \alpha_i\rangle$. But from Lemma 2, $n_p = \operatorname{Max}\{\langle \mu, \alpha_p\rangle|\, \mu \in \Pi\}$. Therefore $q_p > n_p$, which contradicts $q_p < n_p$. This shows that the Kostant region of uniform decomposition is precisely $\{\tau_0 + \gamma|\gamma \in \Lambda^+\}$.

LEMMA 4. $n_i = \operatorname{Max}\{\langle \mu, \sigma\alpha_i\rangle|\, \mu \in \Pi \cap \Lambda^+, \sigma \in W\}$.

PROOF. From Lemma 2, $n_i = \operatorname{Max}\{\langle \mu, \alpha_i\rangle|\, \mu \in \Pi\}$. Every $\mu \in \Pi$ is conjugate to some dominant weight in Π, giving the lemma.

We can now give the sharper result.

LEMMA 5. *$n_i = \langle \lambda, \theta_i\alpha_i\rangle$ where $\theta_i \in W$ is such that $\theta_i\alpha_i$ is the highest root conjugate to α_i. Thus, for L simple, if α_i is a short root, $\theta_i\alpha_i$ is the highest short root, and if α_i is a long root, $\theta_i\alpha_i$ is the highest long root.*

PROOF. Fix $\mu \in \Pi \cap \Lambda^+$. Then for any $\sigma \in W$, $\langle \mu, \sigma\alpha_i\rangle = 2(\mu, \sigma\alpha_i)/(\alpha_i, a_i)$ and $\langle \mu, \theta_i\alpha_i\rangle - \langle \mu, \sigma\alpha_i\rangle = 2(\mu, \theta_i\alpha_i - \sigma\alpha_i)/(\alpha_i, \alpha_i) \geqslant 0$ since μ is dominant and $\theta_i\alpha_i - \sigma\alpha_i$ is a nonnegative sum of positive roots. So $n_i = \operatorname{Max}\{\langle \mu, \theta_i\alpha_i\rangle|\, \mu \in \Pi \cap \Lambda^+\}$. It is a well-known fact that $\theta_i\alpha_i$ is dominant, and since $\lambda - \mu$ is a nonnegative sum of positive roots, we have $\langle \lambda, \theta_i\alpha_i\rangle - \langle \mu, \theta_i\alpha_i\rangle = \langle \lambda - \mu, \theta_i\alpha_i\rangle \geqslant 0$. This says the maximum is attained at $\langle \lambda, \theta_i\alpha_i\rangle$.

This precise characterization of n_i allows us to calculate the l-tuple, $(n_1, \ldots, n_l)$, for each type of algebra in terms of $\lambda = \Sigma_{i=1}^{l} m_i\omega_i$. I have labeled the Dynkin diagrams as in [2]. The results are:

A_l: $n_i = m_1 + m_2 + \cdots + m_l$ for $1 \leqslant i \leqslant l$,

B_l: $n_i = m_1 + 2m_2 + \cdots + 2m_{l-1} + m_l$ for $1 \leqslant i \leqslant l - 1$,

$n_l = 2m_1 + 2m_2 + \cdots + 2m_{l-1} + m_l$,

C_l: $n_i = m_1 + 2m_2 + \cdots + 2m_{l-1} + 2m_l$ for $1 \leqslant i \leqslant l - 1$,

$n_l = m_1 + m_2 + \cdots + m_{l-1} + m_l$,

D_l: $n_i = m_1 + 2m_2 + \cdots + 2m_{l-2} + m_{l-1} + m_l$ for $1 \leqslant i \leqslant l$,

E_6: $n_i = m_1 + 2m_2 + 2m_3 + 3m_4 + 2m_5 + m_6$ for $1 \leqslant i \leqslant 6$,

E_7: $n_i = 2m_1 + 2m_2 + 3m_3 + 4m_4 + 3m_5 + 2m_6 + m_7$ for $1 \leqslant i \leqslant 7$,

$$E_8\colon n_i = 2m_1 + 3m_2 + 4m_3 + 6m_4 + 5m_5 + 4m_6 + 3m_7 + 2m_8 \quad \text{for } 1 \leqslant i \leqslant 8,$$

$$F_4\colon n_1 = n_2 = 2m_1 + 3m_2 + 2m_3 + m_4,$$

$$n_3 = n_4 = 2m_1 + 4m_2 + 3m_3 + 2m_4,$$

$$G_2\colon n_1 = 2m_1 + 3m_2,$$

$$n_2 = m_1 + 2m_2.$$

For L semisimple, these formulas are applied to each simple component separately. If α_i is in a certain component of the Dynkin diagram of L, then the highest root conjugate to α_i involves only the roots in that component. So n_i is calculated according to the type of that component and is given by one of the above formulas involving only those m_j such that α_j is in that component.

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