ON DARBOUX'S LEMMA

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ABSTRACT. We resurrect Carathéodory's proof of Darboux's lemma (normal form $\sum dp_i dq_i$ for a nondegenerate closed 2-form). The proof is simple and straightforward (after eliminating unnecessary steps) and deserves to be better known. It involves, not surprisingly, the canonical differential equations of Hamilton-Jacobi theory.

Darboux's lemma (a closed nondegenerate 2-form can be written locally as $\sum_1^n dp_i dq_i$, in suitable coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$) plays a role in the theory of Hamiltonian systems and symplectic manifolds. The purpose of the following lines is to call attention to Carathéodory's proof in his book [1]. A somewhat more general proposition (Satz 2, p. 135 = Theorem 2, p. 124 in [1]) needs minor modifications, because of some imprecision in the notion of dependence of functions. The special case of Darboux's lemma, however, works well. We describe (and simplify, replacing Carathéodory's reference to the theory of solutions of first order PDE's via characteristics by the lemma below) the proof:

We are in a neighborhood U of O in $\mathbf{R}^{2n}$. We are given a smooth (C^∞) closed 2-form Ω in U , of rank $2n$ (i.e., $\Omega^n = \Omega \wedge \dots \wedge \Omega \neq 0$ in U). We are trying to find new coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ (in a smaller neighborhood V of O) such that Ω takes the form $\sum_1^n dp_i dq_i$ (i.e., $\sum_1^n dp_i \wedge dq_i$).

Since Ω is closed, we can write it as $d\omega$, with a smooth 1-form ω in some neighborhood of O (the Poincaré lemma). Of course ω is determined only up to the differential of a function; this will be crucial for the proof. We write ω as $\sum_1^N g_i df_i$, where the f_i and g_i are functions; then Ω is $\sum_1^N dg_i \wedge df_i$. We plan to use downward induction on N . Clearly for some n of the terms $dg_i \wedge df_i$ the 2-form given by their sum must be nondegenerate at O (since Ω^n is not 0); so that the pertinent functions $(f_{i_1}, \dots, f_{i_n}, g_{i_1}, \dots, g_{i_n})$ are independent at O , have nonzero Jacobian, and can be used as coordinates. Thus, if $N = n$, we are done. If $N > n$, then—as just noted—we can find new coordinates $(u_1, \dots, u_n, v_1, \dots, v_k)$ (with O going to O), so that ω appears in the form

$$\omega = \sum_1^n v_i du_i + \sum_1^k g_r df_r \quad (1)$$

with $N = n + k$, where the f_r and g_r are functions of the u_i and v_i .

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The crux of the argument now is the following proposition, which is in fact the central theme of the classical theory of canonical transformations. We write H (in honor of Hamilton) for the function $-g_1$ of (1).

PROPOSITION. *There exist a change of coordinates of the form*

$$\begin{aligned} u_i &= \varphi_i(a_j, b_j, t), & a_j &= \alpha_j(u_i, v_i, t), \\ v_i &= \psi_i(a_j, b_j, t) \quad \text{or} \quad b_j = \beta_j(u_i, v_i, t), & (i, j &= 1, \dots, n) \\ t &= t, & t &= t \end{aligned} \quad (2)$$

from a neighborhood of O in (u_i, v_i, t) -space to one in (a_j, b_j, t) -space, and a function S of (u_i, v_i, t) or (a_j, b_j, t) such that by virtue of (2) the relation

$$\sum v_i du_i - H dt + dS = \sum b_j da_j \quad (3)$$

holds identically.

(In other words: One can reduce the Hamiltonian to 0, at the expense of a dS -term.)

PROOF. Heuristically, by invariance of the canonical differential equations attached to a Hamiltonian (see (4) below) and the fact that the Hamiltonian on the right of (3) is 0, the a_j and b_j are "constants of the motion", i.e., constant along the trajectories of the canonical equations. We turn this around: Let

$$\dot{u}_i = H_{v_i}, \quad \dot{v}_i = -H_{u_i} \quad (4)$$

be the canonical differential equations for H , and let a_j and b_j be the initial values of the solutions, so that these solutions appear in the form (2) (left half) with

$$\varphi_i(a_j, b_j, 0) = a_i, \quad \psi_i(a_j, b_j, 0) = b_i. \quad (5)$$

The system (2) so defined is clearly invertible near O , with solutions α_j, β_j as in (2). We also define a function $S(a_j, b_j, t)$ by

$$\begin{aligned} \frac{\partial S}{\partial t} &= H(\varphi_i, \psi_i) - \sum \psi_i H_{v_i}(\varphi_k, \psi_k) = H(\varphi_i, \psi_i) - \sum \psi_i \frac{\partial \varphi_i}{\partial t}; \\ S(a_j, b_j, 0) &= 0. \end{aligned} \quad (6)$$

LEMMA. *The form $\sum \psi_i d\varphi_i - H dt + dS$ (in (a_j, b_j, t) -space) is independent of t , i.e., comes via the natural projection from a form in (a_j, b_j) -space.*

PROOF OF LEMMA. We write the form out as

$$\left(\frac{\partial S}{\partial t} - H + \sum \psi_i \frac{\partial \varphi_i}{\partial t} \right) dt + \sum_1^{2n} \left(\frac{\partial S}{\partial c_j} + \sum_i \psi_i \frac{\partial \varphi_i}{\partial c_j} \right) dc_j,$$

where the c_j run through the a_j and b_j . The coefficient of dt vanishes by (6). The t -derivative of the coefficient of dc_j is

$$\frac{\partial}{\partial c_j} \left(\frac{\partial S}{\partial t} \right) + \sum_i \frac{\partial \psi_i}{\partial t} \cdot \frac{\partial \varphi_i}{\partial c_j} + \sum_i \psi_i \frac{\partial}{\partial c_j} \left(\frac{\partial \varphi_i}{\partial t} \right);$$

this vanishes by (4) and (6), as is easily verified. This proves the lemma.

Thus the 1-form of the lemma equals its value for $t = 0$; by (5) and (6) this becomes

$$\sum \psi_i d\varphi_i - H dt + dS = \sum b_j da_j, \quad (7)$$

as identity in (a_j, b_j, t) -space. The proposition is proved.

Expressing (7) via (2) in the (u_i, v_i, t) coordinates, and writing g_1 for $-H$ again, we get the identity

$$\sum v_i du_i + g_1 dt + dS = \sum \beta_j d\alpha_j. \quad (8)$$

In this relation we substitute $f_1(u_i, v_i)$ from (1) for t (we may assume that f_1 is 0 at O so that the substitution is legitimate). Writing S', A_j, B_j for the functions of (u_i, v_i) obtained from S, α_j, β_j by this process, we get the identity

$$\sum_1^n v_i du_i + g_1 df_1 + dS' = \sum_1^n B_j dA_j. \quad (9)$$

Comparing this with (1), and introducing the form $\omega' = \omega + dS'$ (which, as noted in the beginning, serves just as well as ω), we find

$$\omega' = \sum_1^n B_j dA_j + \sum_2^k g_r df_r = \sum_1^{N-1} g'_i df'_i. \quad (10)$$

We have decreased the number of terms by 1; this is the induction step that finishes the proof of Darboux's lemma.

REMARK. Suppose we could use the u_i and a_i as coordinates (i.e., suppose the equations $u_i = \varphi_i(a_j, b_j, t)$ of (2) could be solved for the b_j). Then (3) would mean that the function $S = S(u_i, a_i, t)$ is a solution of the Hamilton-Jacobi equation $S_t(u_i, a_i, t) + H(u_i, S_{u_i}(u_j, a_j, t)) = 0$ (and $b_j = S_{a_j}$). Thus S is a "complete integral" of the H-J equation. (The usual condition $\det(S_{u_i a_j}) \neq 0$ would be automatic.) This is the classical principle that knowledge of a complete integral is "equivalent" to the knowledge of all solutions of the canonical differential equations. (In the present case it would be more natural, because of (5), to use the u_i, b_j and t as coordinates, with S replaced by $S - \sum a_i b_i$.)

REFERENCES

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