MARKOV PROPERTY OF EXTREMAL LOCAL FIELDS

N. DANG-NGOC¹ AND G. ROYER²

ABSTRACT. We show that extremal local field on $(E, \mathcal{E})^T$, with $T = \mathbf{Z}$ or $\mathbf{R}$ and $(E, \mathcal{E})$ standard, possesses the Markov property. This result generalizes that of F. Spitzer in the case $T = \mathbf{Z}$, E countable and a result of G. Royer and M. Yor on extremal measures associated to certain diffusion processes.

I. Introduction. Let $T = \mathbf{R}$ or $\mathbf{Z}$, let $(E, \mathcal{E})$ be a standard Borel space, $(\Omega, \mathcal{Q}, P)$ a probability space and $(X_t)_{t \in T}$ a stochastic process taking values in E . If Λ is a subset of T , we note $\mathcal{Q}_\Lambda$ the sub- σ -algebra of $\mathcal{Q}$ generated by $X_t, t \in \Lambda$.

The process (X_t) is called a *local Markov process* (or *local Markov field*) if for every $a < b$ in T , every positive $\mathcal{Q}_{[a,b]}$ -measurable function f we have

$$(0) \quad E[f | \mathcal{Q}_{]-\infty, a] \cup [b, +\infty[}] = E[f | \mathcal{Q}_{\{a,b\}}] \quad P\text{-a.s.}$$

The process (X_t) is called a *Markov process* (or *Markov field*) if for every t in T , every positive $\mathcal{Q}_{[t, +\infty[}$ measurable function f we have

$$(1) \quad E[f | \mathcal{Q}_{]-\infty, t]} = E[f | \mathcal{Q}_t] \quad P\text{-a.s.}$$

The property (0) is called *local Markov property*, the property (1) is called *Markov property*, it is known that Markov property implies local Markov property (see for example [1]) but the converse is false (cf. [4], [5]).

The purpose of this paper is to show that under quite general conditions the local Markov property and the hypothesis $\mathcal{Q}_\infty = \{\emptyset, \Omega\}$ P -a.s. imply the Markov property, where $\mathcal{Q}_\infty = \bigcap_{t \in T} \mathcal{Q}_{[t, \infty[}$ is the asymptotic σ -algebra of the process (we note similarly $\mathcal{Q}_{-\infty} = \bigcap_{t \in T} \mathcal{Q}_{]-\infty, t]}$).

This result has been proved under the hypotheses E countable and $T = \mathbf{Z}$ by F. Spitzer (cf. [5]). In the case $E = \mathbf{R}$, $T = \mathbf{R}$, G. Royer and M. Yor have proved this result for certain diffusion processes associated to one-dimensional quantum fields (cf. [4]).

II. Sufficient conditions for Markov property.

NOTATIONS. We take the path space $(\Omega, \mathcal{Q}) = (E, \mathcal{E})^T$, $X_t(\omega) = \omega_t, \forall \omega \in E^T, \forall t \in T$. If μ is a probability on $(\Omega, \mathcal{Q})$, and $\Lambda \subset T$, we note μ_Λ the

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restriction of μ on $\mathcal{G}_\Lambda$ and $E_\mu[g|\mathcal{G}_\Lambda]$ the conditional expectation of a random variable g on Ω with respect to the sub- σ -algebra $\mathcal{G}_\Lambda$.

If a probability ν on a σ -algebra $\mathcal{B}$ is absolutely continuous with respect to another probability ν' on $\mathcal{B}$, we write $\nu \ll \nu'$.

We say that μ is a *local Markov field* (resp. *Markov field*) if (X_t) is a local Markov process (resp. Markov process) under μ .

The main result of the paper is

THEOREM 1. *Let $T = \mathbf{R}$ or $\mathbf{Z}$, $(E, \mathcal{G})$ a standard Borel space. Let μ be a local Markov field on $(\Omega, \mathcal{G})$ such that*

(i) *For every $t \in T$, there exist $t' < t < t''$ in T and σ -finite measures $\nu_{t'}$, ν_t , $\nu_{t''}$ on $(E, \mathcal{G})$ such that*

$$\mu_{t',t,t''} \ll \nu_{t'} \otimes \nu_t \otimes \nu_{t''}.$$

(ii) $\mathcal{G}_\infty$ (resp. $\mathcal{G}_{-\infty}$) is trivial for μ .

Then μ is a Markov process.

We give the proof for the case $T = \mathbf{R}$ (the case $T = \mathbf{Z}$ is similar) under the hypothesis $\mathcal{G}_\infty$ trivial (for the case $\mathcal{G}_{-\infty}$ trivial, we use the additional remark that the Markov property is symmetric with respect to the direction of time).

PROOF. Let $t \in T$, $t' < t < t''$, $\nu_{t'}$, ν_t , $\nu_{t''}$ as in (i), let k be a positive integer greater than t'' , $S = \{t', t, t'', k, k+1, k+2, \dots\}$

(a) On $(E, \mathcal{G})^S = (E, \mathcal{G})^{t'} \times (E, \mathcal{G})^{S \setminus t'}$ we have $\mu_S \ll \nu_{t'} \otimes \mu_{S \setminus t'}$:

Let $(E, \mathcal{G})^S = (E, \mathcal{G})^{(t')} \times (E, \mathcal{G})^{(t',t'')} \times (E, \mathcal{G})^{(k,k+1,\dots)}$,

$$(E, \mathcal{G})^S = (X, \mathcal{X}) \times (Y, \mathcal{Y}) \times (Z, \mathcal{Z}).$$

Consider the measure $\mu_{\{t',t,t''\}}$ on $X \times Y = E^{(t')} \times E^{(t',t'')}$ and let $\mu_{(x_{t'},x_{t''})}(F)$, for $(x_{t'}, x_{t''}) \in Y$, $F \in \mathcal{X}$ be the transition probability on $Y \times \mathcal{X}$ such that

$$(2) \quad \mu_{\{t',t,t''\}} = \int_Y \mu_{(x_{t'},x_{t''})} d\mu_{\{t',t''\}}(x_{t'}, x_{t''})$$

(cf. [3, Proposition V.4.4, p. 183]).

And similarly on $X \times (Y \times Z)$ let $\mu_{(x_{t'},x_{t''},x_k,x_{k+1},\dots)}(F)$, for $(x_{t'}, x_{t''}, x_k, x_{k+1}, \dots) \in Y \times Z$, $F \in \mathcal{X}$ be the transition probability on $(Y \times Z) \times \mathcal{X}$ such that

$$(3) \quad \mu_S = \int_{Y \times Z} \mu_{(x_{t'},x_{t''},x_k,x_{k+1},\dots)} d\mu_{S \setminus t'}(x_{t'}, x_{t''}, x_k, \dots).$$

By local Markov property (0), we have

$$(4) \quad \mu_{(x_{t'},x_{t''},x_k,\dots)} = \mu_{(x_{t'},x_{t''})}, \quad \text{for } \mu_{S \setminus t'}\text{-almost all } (x_{t'}, x_{t''}).$$

But by (i), we have $\mu_{t',t,t''} \ll \nu_{t'} \otimes \nu_t \otimes \nu_{t''}$ and $\mu_{t',t''} \ll \nu_{t'} \otimes \nu_{t''}$, so relation (2) implies

$$(5) \quad \mu_{(x_{t'},x_{t''})} \ll \nu_t, \quad \text{for } \mu_{t',t''}\text{-almost all } (x_{t'}, x_{t''}).$$

Comparing (4) and (5), we deduce that

$$\mu_{(x_{t'},x_{t''},x_k,\dots)} \ll \nu_t, \quad \text{for } \mu_{S \setminus t'}\text{-almost all } (x_{t'}, x_{t''}, x_k, \dots).$$

Combining this relation and (3), we obtain

$$\mu_S \ll \nu_t \otimes \mu_{S \setminus t}.$$

(b) We have $\mathcal{Q}_t = \lim_{n \rightarrow \infty} \mathcal{Q}_t \vee (\bigvee_{m \geq n} \mathcal{Q}_m)$: Let $(X, \mathcal{X}) = (E, \mathcal{E})^{(t)}$, $(W, \mathcal{W}) = (E, \mathcal{E})^{S \setminus \{t\}}$, then $(E, \mathcal{E})^S = (X, \mathcal{X}) \times (W, \mathcal{W})$.

Let $\mathcal{B}_n = \bigvee_{m \geq n} \mathcal{Q}_m$, for $n \geq k$, we shall show that $\lim_{n \rightarrow \infty} \mathcal{Q}_t \vee \mathcal{B}_n = \mathcal{Q}_t$, μ_S -a.e. But we remark that $-\mathcal{Q}_t = \mathcal{X}$, $-\mathcal{B}_n \subset \mathcal{W}$, $\forall n \geq k$ and $\lim_{n \rightarrow \infty} \mathcal{B}_n = \{\emptyset, \Omega\}$, $\mu_{S \setminus t}$ -a.e. $-\mathcal{X}$ and $\mathcal{W}$ are independent with respect to $\nu_t \otimes \mu_{S \setminus t}$. Therefore

$$\lim_n \mathcal{Q}_t \vee \mathcal{B}_n = \mathcal{Q}_t, \quad \nu_t \otimes \mu_{S \setminus t}\text{-a.e.}$$

and also μ_S -a.e. since $\mu_S \ll \nu_t \otimes \mu_{S \setminus t}$.

(c) Now let $f \geq 0$, $\mathcal{Q}_{[\alpha, \beta]}$ -measurable with $t \leq \alpha \leq \beta$, then (1) is satisfied:

Let $n > \beta$ with n positive integer, by local Markov property (0), we have

$$\begin{aligned} E_\mu[f|\mathcal{Q}_{[-\infty, t] \cup [n, +\infty]}] &= E_\mu[f|\mathcal{Q}_{(t, n)}] \\ &= E_\mu[f|\mathcal{Q}_t \vee \mathcal{Q}_{(n, n+1, \dots)}]. \end{aligned}$$

So by (b) and martingale theorem, we have

$$\lim_{n \rightarrow \infty} E_\mu[f|\mathcal{Q}_{[-\infty, t] \cup [n, +\infty]}] \rightarrow E_\mu[f|\mathcal{Q}_t].$$

Taking the conditional expectation with respect to $\mathcal{Q}_{[-\infty, t]}$ of the both sides, we obtain

$$E_\mu[f|\mathcal{Q}_{[-\infty, t]}] = E[f|\mathcal{Q}_t], \quad \mu\text{-a.e.}$$

(d) From (c) we deduce (1) since the $\mathcal{Q}_{[\alpha, \beta]}$, $t \leq \alpha \leq \beta$, generate $\mathcal{Q}_{[t, \infty]}$.

REMARK. It is well known that the conditions $\mathcal{B}_n \searrow \{\emptyset, \Omega\}$ do not imply $\mathcal{Q}_t \vee \mathcal{B}_n \searrow \mathcal{Q}_t$ without extra hypotheses.

III. Application to statistical mechanics and one dimensional field theory.

LOCAL MARKOV SPECIFICATIONS. Let $T = \mathbf{R}$ or $\mathbf{Z}$, $(E, \mathcal{E})$ be a standard Borel space, $(\Omega, \mathcal{Q}) = (E, \mathcal{E})^T$.

We call local Markov specifications (see [2]) the given of a family $\pi = (\pi_{[a, b]})_{a < b}$ of transition probabilities $\pi_{[a, b]}(\omega, A)$ ($\omega \in \Omega$, $A \in \mathcal{Q}$) such that

(i) For every $t \in T$ there exist $a < t < b$, a σ -finite measure ν_t on $(E, \mathcal{E})$ such that the restriction of $\pi_{[a, b]}(\omega, \cdot)$ to $\mathcal{Q}_t$ is absolutely continuous with respect to ν_t , $\forall \omega \in \Omega$.

(ii) $\pi_{[a, b]}(\cdot, A)$ is $\mathcal{Q}_{[-\infty, a] \cup [b, +\infty]}$ -measurable, $\forall A \in \mathcal{Q}$.

$\pi_{[a, b]}(\cdot, A)$ is $\mathcal{Q}_{a, b}$ -measurable, for $A \in \mathcal{Q}_{[a, b]}$.

$\pi_{[a, b]}(\cdot, A) = 1_A(\cdot)$ for $A \in \mathcal{Q}_{[-\infty, a] \cup [b, +\infty]}$.

(iii) If $[a, b] \subset [c, d]$, we have $\pi_{[c, d]}\pi_{[a, b]} = \pi_{[c, d]}$ where

$$\pi_{[c, d]}\pi_{[a, b]}(\omega, A) = \int_\Omega \pi_{[c, d]}(\omega, d\omega') \pi_{[a, b]}(\omega', A), \quad \forall A \in \mathcal{Q}, \forall \omega \in \Omega.$$

We call *local Markov field* (or *Gibbs state*) specified by π every probability μ on $(\Omega, \mathcal{Q})$ satisfying

$$E_\mu[A|\mathcal{Q}_{[-\infty, a] \cup [b, +\infty]}](\omega) = \pi_{[a, b]}(\omega, A), \quad \mu\text{-a.e.}, \forall A \in \mathcal{Q}_{[a, b]}.$$

The set $\mathcal{G}(\pi)$ of all local Markov fields specified by π is a convex set which is possibly empty.

A local Markov field $\mu \in \mathcal{G}(\pi)$ is *extremal* iff

$$\bigcap_n \bigvee_{|I| > n} \mathcal{Q}_I = \{\emptyset, \Omega\}, \quad \mu\text{-a.e. (cf. [2])}.$$

Theorem 1 implies

THEOREM 2. *Let $(\pi_{[a,b]})_{a < b}$ be local Markov specifications. Every extremal point of $\mathcal{G}(\pi)$ has the Markov property.*

REMARKS. (1) When $T = \mathbb{Z}$, E countable, this result implies Theorem 6 of F. Spitzer (cf. [5]).

(2) This result also implies Theorem 3.9 of G. Royer and M. Yor (cf. [4]).

(3) It should be interesting to know whether similar results hold when $T = \mathbb{Z}^d$ or $\mathbb{R}^d$ with $d \geq 2$.

(4) The characterization of Gibbs states of $\mathcal{G}(\pi)$ possessing the Markov property (other than extremal Gibbs states) is given in [1].

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