ANOTHER EXTREMAL PROPERTY OF PERFECT SPLINES

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ABSTRACT. Let $t = \{t_i\}$, $i = 1, 2, \dots, n + k$, be a given sequence in $[a, b]$, $f_0 \in W_\infty^k[a, b]$, $A > \|f_0^{(k)}\|_\infty$, and $F = \{f \in W_\infty^k[a, b]: f|_t = f_0|_t \text{ and } \|f^{(k)}\|_\infty < A\}$. We show that F contains precisely two perfect splines g, h of degree k with $|g^{(k)}| = |h^{(k)}| = A$ and n interior nodes, and for all $f \in F$, $\min(g(x), h(x)) < f(x) < \max(g(x), h(x)); \forall x \in [a, b]$.

1. Introduction. Let t denote a sequence $a = t_1 \leq t_2 \leq \dots \leq t_{n+k} = b$ ($n, k \geq 1$) such that $t_i < t_{i+k}$ for all $i = 1, 2, \dots, n$, and define

$$W_\infty^k[a, b] = \{f \in C^{k-1}[a, b]: f^{(k-1)} \text{ absolutely continuous and } f^{(k)} \in L_\infty[a, b]\}. \quad (1.1)$$

Take a function $f_0 \in W_\infty^k[a, b]$, and set

$$F(t, f_0) = \{f \in W_\infty^k[a, b]: f|_t = f_0|_t\}. \quad (1.2)$$

Recently De Boor [1] and Karlin [2] showed the existence of a perfect spline $p \in F(t, f_0)$ of degree k with less than n interior knots, i.e. a function of the form

$$p(x) = \sum_{i=0}^{k-1} a_i x^i + c \left[x^k + 2 \sum_{i=1}^{r-1} (-1)^i (x - \xi_i)_+^k \right] \quad (1.3)$$

for some real constants $a_0, a_1, \dots, a_{k-1}, c$ and for $a < \xi_1 < \dots < \xi_{r-1} < b$ ($r \leq n$), and that this perfect spline is a solution of the problem of minimising $\|f^{(k)}\|_\infty$ over all f in $F(t, f_0)$. A special case of this result was earlier considered by Schoenberg [7], who attributed his main results to Glaeser [3] and Louboutin [5].

Now, let A be a positive number larger than $\|f_0^{(k)}\|_\infty$, and let

$$F \equiv F(t, f_0, A) = \{f \in F(t, f_0): \|f^{(k)}\|_\infty < A\}. \quad (1.4)$$

Our main result is the following, which generalises some work of McClure [6].

THEOREM 1. *There are precisely two perfect splines g, h in F with $|g^{(k)}| = |h^{(k)}| = A$ and less than $(n + 1)$ interior nodes. For any $f \in F$*

$$\min(g(x), h(x)) \leq f(x) \leq \max(g(x), h(x)), \quad \forall x \in [a, b]. \quad (1.5)$$

Received by the editors April 12, 1977.

AMS (MOS) subject classifications (1970). Primary 41A05, 41A15; Secondary 26A87.

Key words and phrases. Perfect splines.

Furthermore g (or h) has exactly n interior nodes $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ which are the unique set of points such that

$$\begin{aligned} \psi(a) - 2\psi(\alpha_1) + 2\psi(\alpha_2) - \cdots + 2(-1)^n \psi(\alpha_n) + (-1)^{n+1} \psi(b) \\ = A^{-1} \int_a^b f_0^{(k)} \psi' \quad \left(\text{or } -A^{-1} \int_a^b f_0^{(k)} \psi' \right) \end{aligned} \quad (1.6)$$

for all ψ with $\psi' \in V(\mathbf{t})$.

Here $V(\mathbf{t})$ denotes the class of all spline functions of degree $k-1$, with knots at $\mathbf{t}$, which vanish outside $[a, b]$.

The above result persists when the derivatives are replaced by a differential operator of the form

$$L_k f = \left(\frac{d}{dx} \frac{1}{W_k(x)} \frac{d}{dx} \frac{1}{W_{k-1}(x)} \cdots \frac{d}{dx} \frac{1}{W_1(x)} \right) f, \quad (1.7)$$

where $W_i(x) \in C^k[a, b]$ and $W_i(x) > 0$, $i = 1, 2, \dots, k$, on $[a, b]$. A function $f \in W_\infty^k[a, b]$ is called a *perfect L_k -spline* if $|L_k f| = c$, c constant.

Let $B_i(x)$, $i = 1, 2, \dots, n$ be the B -splines associated with the differential operator L_k , with nodes $t_i, t_{i+1}, \dots, t_{i+k}$, and let $V(L_k, \mathbf{t})$ be the space generated by $B_i(x)$, $i = 1, 2, \dots, n$. Then $V(L_k, \mathbf{t})$ consists of functions $S(x)$ which vanish outside $[a, b]$ and such that $(L_k S)(x) = 0$ whenever $x \notin \mathbf{t}$.

Now, let A be a positive number larger than $\|L_k f_0\|_\infty$ and let

$$F(L_k, \mathbf{t}, f_0, A) = \{f \in W_\infty^k[a, b] : f|_{\mathbf{t}} = f_0|_{\mathbf{t}} \text{ and } \|L_k f\|_\infty \leq A\}. \quad (1.8)$$

Then the following result holds:

THEOREM 2. *There are precisely two perfect L_k -splines g, h in $F(L_k, \mathbf{t}, f_0, A)$ with $|L_k g| = |L_k h| = A$ and less than $n+1$ interior nodes. For any $f \in F(L_k, \mathbf{t}, f_0, A)$,*

$$\min(g(x), h(x)) \leq f(x) \leq \max(g(x), h(x)) \quad \forall x \in [a, b]. \quad (1.9)$$

Furthermore g (or h) has exactly n interior nodes $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ which are the unique set of points such that

$$\begin{aligned} \psi(a) - 2\psi(\alpha_1) + 2\psi(\alpha_2) - \cdots + 2(-1)^n \psi(\alpha_n) + (-1)^{n+1} \psi(b) \\ = A^{-1} \int_a^b L_k f_0 \psi' \quad \left(\text{or } -A^{-1} \int_a^b L_k f_0 \psi' \right) \end{aligned} \quad (1.10)$$

for all ψ with $\psi' \in V(L_k, \mathbf{t})$.

Our proof of Theorem 1 in §3 depends on an extremal problem in Banach space, stated in §2, combined with an argument of De Boor [1].

2. Auxiliary results. Our main result depends on Theorem 3 below, which is a straightforward deduction from the Hahn-Banach Extension Theorem. A proof and some further applications were given in [4].

THEOREM 3. *Suppose V is a closed subspace of a real normed space W and $k \in W - V$. Suppose $\phi_0 \in W^*$, the continuous dual of W , and $\|\phi_0\| \leq A$.*

Then for $w \in V$,

$$A\|k + w\| - \phi_0(k + w) = \inf_{v \in V} \{A\|k + v\| - \phi_0(k + v)\} \quad (2.1)$$

if and only if $\exists \phi \in W^*$ with

$$\phi|_V = \phi_0|_V, \quad \|\phi\| = A \quad \text{and} \quad \phi(k + w) = A\|k + w\|. \quad (2.2)$$

COROLLARY 4. Let X be a σ -finite measure space, V a closed subspace of $L^1(X)$ and $k \in L^1(X) - V$. Suppose $\phi_0 \in L^\infty(X)$ and $\|\phi_0\| \leq A$. Then a necessary and sufficient condition that for $w \in W$,

$$A\|k + w\|_1 - \int (k + w)\phi_0 = \inf_{v \in V} \left\{ A\|k + v\|_1 - \int (k + v)\phi_0 \right\} \quad (2.3)$$

is that $\exists \phi \in L^\infty(X)$ with $\int \phi v = \int \phi_0 v, \forall v \in V$, and

$$\|\phi\|_\infty = A \quad \text{and} \quad \int (k + w)\phi = A\|k + w\|_1. \quad (2.4)$$

REMARK. Observe that (2.4) is equivalent

$$\phi(k + w) = A|k + w|. \quad (2.5)$$

If the set of zeros of $(k + w)$ has measure zero, then (2.5) is equivalent to

$$\phi = A \operatorname{sgn}(k + w). \quad (2.6)$$

We shall also require the following simple property of spline functions.

LEMMA 5. If $\alpha_1, \alpha_2, \dots, \alpha_r$ are distinct points in (a, b) and $r < n$, then $\exists$ a nonzero $\phi \in V(\mathbf{t})$ which changes sign precisely at $\alpha_1, \alpha_2, \dots, \alpha_r$ (0 can be taken as + or -).

PROOF. We fix $k \geq 1$ and prove by induction on n . If $n = 1$, any nonzero element of $V(\mathbf{t})$ has no changes of sign and so the result is true.

Suppose the result is true for $n < N$ and take $\mathbf{t}$ with $n = N$ and $\alpha_1, \alpha_2, \dots, \alpha_r$ with $r < N$. If $r < N - 1$, we may apply the hypothesis to $t_2 \leq t_3 \leq \dots \leq t_{N+k}$ or $t_1 \leq t_2 \leq \dots \leq t_{N+k-1}$ to obtain $\phi \in V(\mathbf{t})$ which changes sign precisely at $\alpha_1, \alpha_2, \dots, \alpha_r$. Suppose then that $r = N - 1$. Let $B_1, B_2, \dots, B_N$ be the B -splines for $\mathbf{t}$. Choose $a_1, a_2, \dots, a_N$, not all zero such that

$$a_1 B_1(\alpha_i) + a_2 B_2(\alpha_i) + \dots + a_N B_N(\alpha_i) = 0, \quad \forall i = 1, 2, \dots, N - 1, \quad (2.7)$$

and let

$$\phi = a_1 B_1 + a_2 B_2 + \dots + a_N B_N. \quad (2.8)$$

First suppose ϕ is oscillating in $[t_1, t_{N+k}]$. Since ϕ has at most $N - 1$ zeros in (t_1, t_{N+k}) , $\alpha_1, \alpha_2, \dots, \alpha_{N-1}$ are simple zeros of ϕ and so ϕ changes sign precisely at these points. On the other hand if ϕ vanishes on at least one interval in $[t_1, t_{N+k}]$, then in each segment $[t_i, t_j]$ on which ϕ is oscillating, we must have $j - i \geq k$ and ϕ has at most $(j - i - k)$ zeros. Hence $[t_i, t_j]$ contains at most $(j - i - k)$ points of the set $\{\alpha_1, \alpha_2, \dots, \alpha_{N-1}\}$. We can

then apply the inductive hypothesis to each of these segments to obtain the required ϕ . $\square$

3. Proof of Theorem 1. For $g \in L_\infty[a, b]$,

$$\int_a^b \phi g = \int_a^b \phi f_0^{(k)} \quad \forall \phi \in V(t)$$

if and only if $\exists f \in W_\infty^k[a, b]$ with $f|t = f_0|t$ and $f^{(k)} = g$.

There is a function K on $[a, b] \times [a, b]$ such that for any $f \in F$ and $x \in [a, b]$,

$$f(x) - f_0(x) = \int_a^b K(x, t) \{f^{(k)}(t) - f_0^{(k)}(t)\} dt. \quad (3.1)$$

Hence, $\forall \phi \in V(t)$

$$f(x) - f_0(x) = \int_a^b (K(x, t) + \phi(t)) \{f^{(k)}(t) - f_0^{(k)}(t)\} dt, \quad (3.2)$$

whence

$$f(x) - f_0(x) \leq \inf_{\phi \in V(t)} \left\{ A \int_a^b |K(x, t) + \phi(t)| dt - \int_a^b (K(x, t) + \phi(t)) f_0^{(k)}(t) dt \right\} \quad (3.3)$$

and

$$f(x) - f_0(x) \geq \sup_{\phi \in V(t)} \left\{ -A \int_a^b |K(x, t) + \phi(t)| dt - \int_a^b (K(x, t) + \phi(t)) f_0^{(k)}(t) dt \right\}. \quad (3.4)$$

Since $V(t)$ is finite dimensional, $\exists \phi_1 \in V(t)$ for which the infimum in (3.3) is attained. By Corollary 4, $\exists \gamma \in L_\infty[a, b]$ with $\|\gamma\|_\infty = A$ and $\int_a^b \phi \gamma = \int_a^b \phi f_0^{(k)}$, $\forall \phi \in V$, and

$$\int_a^b \{K(x, t) + \phi_1(t)\} \gamma(t) dt = A \int_a^b |K(x, t) + \phi_1(t)| dt. \quad (3.5)$$

Choose $h_x \in W_\infty^k[a, b]$ with $h_x|t = f_0|t$ and $h_x^{(k)} = \gamma$.

It follows from (3.3) that

$$f(x) \leq h_x(x) \quad \forall f \in F. \quad (3.6)$$

Now $K(x, \cdot) + \phi_1$ is a spline function with nodes at $t \cup \{x\}$, i.e. it is in $V(t \cup \{x\})$. Also from (3.5) we know that

$$h_x^{(k)}(t) = A \operatorname{sgn}(K(x, t) + \phi_1(t)) \quad \text{when } K(x, t) + \phi_1(t) \neq 0. \quad (3.7)$$

Using the method of De Boor in [1], we may choose h_x so that $|h_x^{(k)}| = A$ on (a, b) and $h_x^{(k)}$ changes sign at at most n points in (a, b) , i.e. h_x is a perfect spline with less than $n + 1$ interior nodes.

Similarly, using (3.4), a perfect spline g_x with less than $(n + 1)$ interior nodes can be found such that

$$g_x(x) \leq f(x), \quad \forall f \in F. \quad (3.8)$$

Clearly $g_x(x) < h_x(x)$ if $x \notin t$.

For convenience we shall call $f \in F$ an extremal function in F if $|f^{(k)}| = A$ on (a, b) and f has at most n interior nodes. We now show that if f is an extremal function in F , then it has precisely n interior nodes. For suppose it has less than n interior nodes. By Lemma 5 there is a nonzero $\phi \in V(t)$ which changes sign at the nodes of f , so that $\phi f^{(k)} = A|\phi|$. Hence

$$\int_a^b \phi f^{(k)} = A \int_a^b |\phi| > \left| \int_a^b \phi f_0^{(k)} \right|,$$

which contradicts $f|t = f_0|t$.

Next, suppose that f is an extremal function in F , and for some i , $1 < i \leq n$, $t_i = \dots = t_{i+k-1}$, i.e. t_i is an interior node of t of multiplicity k . We claim that f cannot have a node at t_i . For, if we let $t_1 = \{t_1, t_2, \dots, t_{i+k-1}\}$ and $t_2 = \{t_i, t_{i+1}, \dots, t_{n+k}\}$, and apply the above result to $F(t_1, f_0, A)$ and $F(t_2, f_0, A)$, then f must have at least $i-1$ nodes in (t_1, t_i) and $n-i+1$ nodes in (t_i, t_{n+k}) . Since f has only n nodes in (t_1, t_{n+k}) , it cannot have a node at t_i .

We now show that if f is an extremal function in F , then for any $x \in [a, b]$, $f(x)$ equals $g_x(x)$ or $h_x(x)$. We may suppose $x \notin t$. By Lemma 5, we may choose a nonzero ψ in $V(t \cup \{x\})$ which changes sign at the nodes of f , so that $\psi f^{(k)} = A|\psi|$. Now $\psi = \lambda K(x, \cdot) + \phi$, for some real number λ and $\phi \in V(t)$. By an argument above, $\phi f^{(k)} \neq A|\phi|$ and so $\lambda \neq 0$. Hence $(K(x, \cdot) + \phi) f^{(k)} = \pm A|K(x, \cdot) + \phi|$, and so from (3.2), (3.3), (3.4) $f(x)$ equal $g_x(x)$ or $h_x(x)$.

Now let f_1, f_2 be extremal functions in F . We shall show that if f_1 and f_2 are equal at any point not in t , then they are equal throughout $[a, b]$. Suppose $f_1(x_0) = f_2(x_0)$ for some $x_0 \in (t_{r-1}, t_r)$. We know that for any $x \in (t_{r-1}, t_r)$, $f_i(x) = h_x(x)$ or $f_i(x) = g_x(x)$ ($i = 1, 2$). Since $g_x(x) < h_x(x) \forall x \in (t_{r-1}, t_r)$, then by continuity, $f_1 = f_2$ throughout (t_{r-1}, t_r) , and so $f_1^{(m)}(t_r) = f_2^{(m)}(t_r)$, $m = 0, 1, \dots, k-1$. Next, suppose $f_1 \neq f_2$ on (t_r, t_s) . We may assume that $f_1 < f_2$ on (t_r, t_{r+1}) . For any $f \in F$, $f_1 < f < f_2$ on (t_r, t_{r+1}) and so $f^{(m)}(t_r) = f_1^{(m)}(t_r)$, $m = 0, 1, \dots, k-1$. Hence t_r has multiplicity k , and so neither f_1 nor f_2 can have a node at t_r . Since $f_1 = f_2$ on (t_{r-1}, t_r) , it follows that $f_1 = f_2$ on some nonempty interval (t_r, α) , which contradicts $f_1 \neq f_2$ on (t_r, t_s) . So $f_1 = f_2$ on (t_r, b) and similar argument shows that $f_1 = f_2$ on (a, t_{r-1}) . Hence $f_1 = f_2$ on $[a, b]$.

We know that there are two distinct extremal functions g, h in F . Also if f is any other extremal function in F , then for any $x \in (a, b)$, $x \notin t$, $f(x)$ equals $g(x)$ or $h(x)$, and so $f = g$ or h on $[a, b]$. Furthermore for any $f \in F$ and $x \in [a, b]$,

$$\min(g(x), h(x)) \leq f(x) \leq \max(g(x), h(x)).$$

Finally, we recall that for $g \in L_\infty[a, b]$, $\exists f \in W_\infty^k[a, b]$ with $f|t = f_0|t$ and $f^{(k)} = g$ if and only if

$$\int_a^b \phi g = \int_a^b \phi f_0^{(k)} \quad \forall \phi \in V(t).$$

Then there is a perfect spline f in F with $|f^{(k)}| = A$ and interior nodes at $\alpha_1 < \alpha_2 < \dots < \alpha_n$ if and only if

$$\pm A \left\{ \int_a^{\alpha_1} \phi - \int_{\alpha_1}^{\alpha_2} \phi + \dots + (-1)^n \int_{\alpha_n}^b \phi \right\} = \int_a^b \phi f_0^{(k)}, \quad \forall \phi \in V(t), \quad (3.9)$$

from which (1.6) follows. $\square$

Let $B = \inf\{\|f^{(k)}\|_\infty : f \in F(t, f_0)\}$. Theorem 1 can be applied for all $A > B$. Letting A converge to B we have the following result.

COROLLARY 6. *There are perfect splines g, h in $F(t, f_0)$ with $|g^{(k)}| = |h^{(k)}| = B$ and less than $(n + 1)$ interior nodes such that (1.5) holds for any $f \in F(t, f_0)$ with $\|f^{(k)}\|_\infty = B$.*

In this case g and h may have less than n interior nodes. Indeed g may equal h , so that g is the only function in $F(t, f_0)$ with $\|g^{(k)}\|_\infty = B$, and by [1], g has less than n interior nodes.

As a special case of Theorem 1, take $n = k$, $t_1 = t_2 = \dots = t_n = -1$, $t_{n+1} = t_{n+2} = \dots = t_{2n} = 1$, $f_0 = 0$ and $A > 0$. Then

$$F = \{f \in W_\infty^n[-1, 1] : f^{(r)}(-1) = f^{(r)}(1) = 0, \\ r = 0, 1, \dots, n-1, \text{ and } \|f^{(r)}\|_\infty \leq A\}. \quad (3.10)$$

Theorem 1 tells us there is a perfect spline $h \in F$ with $\|h^{(k)}\|_\infty = A$ and n interior nodes, and $\pm h$ are the only such spline functions in F . Moreover for any $f \in F$,

$$|f(x)| \leq |h(x)|, \quad \forall x \in [-1, 1]. \quad (3.11)$$

The nodes of h are the unique set of points $\alpha_1, \alpha_2, \dots, \alpha_n$ for which

$$\psi(-1) - 2\psi(\alpha_1) + \dots + 2(-1)^n \psi(\alpha_n) + (-1)^{n+1} \psi(1) = 0 \quad (3.12)$$

for all polynomials ψ of degree n . It can be shown (for instance by using Lemma 1 of Schoenberg [7]) that (3.12) is satisfied if $\alpha_\nu = -\cos(\nu\pi/(n+1))$, $\nu = 1, 2, \dots, n$, the zeros of Chebyshev polynomial of the second kind.

Next consider the differential operator $L = D(D^2 + 1) \dots (D^2 + n^2)$ and the sequence t defined by $t_1 = t_2 = \dots = t_{2n+1} = 0$, $t_{2n+2} = t_{2n+3} = \dots = t_{4n+2} = 2\pi$. For $A > 0$, define $\tilde{F} \equiv F(L, t, 0, A) = \{f \in W_\infty^n[0, 2\pi] : f^{(r)}(0) = f^{(r)}(2\pi) = 0, r = 0, 1, \dots, 2n, \text{ and } \|Lf\|_\infty \leq A\}$. The operator L is not of the form (1.7) on $[0, 2\pi]$ and so we cannot apply Theorem 2 directly. However it can be put in the form (1.7) on any proper subinterval of $[0, 2\pi]$.

It follows that there is a perfect L -spline $\tilde{h} \in \tilde{F}$ with $\|L\tilde{h}\|_\infty = A$ and knots at $\nu\pi/(n+1)$, $\nu = 1, 2, \dots, 2n+1$. We can deduce that for any $f \in \tilde{F}$,

$$|f(x)| \leq |\tilde{h}(x)|, \quad \forall x \in [0, 2\pi]. \quad (3.13)$$

By the same argument as in §3, it can be shown that $\pm \tilde{h}$ are the only

perfect L -splines in $\tilde{F}$ with $|L\tilde{h}| = A$ and less than $2n + 2$ interior nodes.

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