CHARACTERIZATIONS OF AMENABLE BANACH ALGEBRAS

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ABSTRACT. In this paper we show that a Banach algebra A is amenable if and only if A has any one of the following properties: (i) whenever X is a Banach A -bimodule and Y is an A -submodule of X , then for each $f \in Y^*$ such that $a \cdot f = f \cdot a$, $a \in A$, there exists $\tilde{f} \in X^*$ which extends f and $a \cdot \tilde{f} = \tilde{f} \cdot a$ for all $a \in A$; (ii) whenever X is a Banach A -bimodule, there exists a bounded projection P from X^* onto $\{f \in X^*; a \cdot f = f \cdot a \text{ for all } a \in A\}$ such that $T \cdot P = P \cdot T$ for any weak* continuous bounded linear operator T from X^* into X^* commuting with the action of A on X^* . The class of ultraweakly amenable von Neumann algebras with separable predual can be similarly characterized.

1. Introduction. In [6], B. Johnson proved that a locally compact group G is amenable if and only if the group algebra $A (= L_1(G))$ of G has the following property:

(J) Whenever X is a Banach A -bimodule, every bounded derivation from A into X^* is inner.

Banach algebras A with property (J) are called *amenable Banach algebras* [6, p. 60]. In this paper we shall extend a result of J. Bunce in [1] and show that a Banach algebra A is amenable if and only if A has certain Hahn Banach extension property. We also show that amenability of A is equivalent to the existence of a bounded projection P from X^* onto the subspace $Z(A, X^*) = \bigcap_{a \in A} \{f \in X^*; a \cdot f = f \cdot a\}$ whenever X is a Banach A -bimodule and P commutes with any weak* continuous bounded linear operator from X^* into X^* which commutes with the action of A on X^* . Ultraweakly amenable von Neumann algebras with separable preduals can be similarly characterized.

Examples of amenable Banach algebras include all commutative C^* -algebras, the group algebra and the group C^* -algebra of an amenable group (see [6]). However, the C^* -algebra generated by left translations on $l_2(G)$, when G is the free group on two generators, is not amenable [3, Proposition 2].

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2. Some notations. If X is a normed linear space, then X^* will denote its continuous dual. Also if $x \in X$ and $f \in X^*$, then the value of f at x will be denoted by $f(x)$ or $\langle f, x \rangle$.

Let A be a Banach algebra. A Banach space X which is also an A -bimodule is called a *Banach A -bimodule* if there exists a constant $K > 0$ such that $\|a \cdot x\| \leq K\|a\| \|x\|$ and $\|x \cdot a\| \leq K\|a\| \|x\|$ for all $a \in A, x \in X$. If X is a Banach A -bimodule, then X^* becomes a Banach A -bimodule with the action of A on X^* defined by $\langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle$ and $\langle f \cdot a, x \rangle = \langle f, a \cdot x \rangle$ for all $a \in A, f \in X^*$ and $x \in X$.

If X is a Banach A -bimodule, then a bounded linear map δ from A into X^* is called a *derivation* if $\delta(a \cdot b) = a \cdot \delta(b) + \delta(a) \cdot b$ for all $a, b \in A$; a derivation $\delta: A \rightarrow X^*$ is *inner* if there exists $f \in X^*$ such that

$$\delta(a) = a \cdot f - f \cdot a \quad \text{for all } a \in A.$$

3. Amenable Banach algebras. If A is a Banach algebra and X is a Banach A -bimodule, write

$$Z(A, X^*) = \bigcap_{a \in A} \{f \in X^*; a \cdot f = f \cdot a\}.$$

Then $Z(A, X^*)$ is a closed linear subspace of X^* which is invariant under each bounded linear operator from X^* into X^* commuting with the action of A .

THEOREM 1. *Let A be a Banach algebra. The following are equivalent:*

- (a) A is amenable.
- (b) For any Banach A -bimodule X and any Banach A -submodule Y of X , each linear functional in $Z(A, Y^*)$ has an extension to a linear functional in $Z(A, X^*)$.
- (c) For any Banach A -bimodule, there exists a bounded projection from X^* onto $Z(A, X^*)$ which commutes with any weak* continuous bounded linear operator from X^* into X^* commuting with the action of A on X^* .

PROOF. (a) $\Rightarrow$ (b) The quotient Banach space X/Y may be regarded as a Banach A -bimodule with the action of A on X/Y defined by $a \cdot \bar{x} = \overline{ax}$ and $x \cdot a = \overline{xa}$ for all $a \in A, x \in X$, where $\bar{x}$ denotes the equivalence class of x in X/Y . Let $f \in Z(A, Y^*)$ and $\tilde{f} \in X^*$ be any extension of f to X . If $a \in A$, then $a \cdot \tilde{f} - \tilde{f} \cdot a \in Y^\perp$, where $Y^\perp = \{f \in X^*; f(y) = 0 \text{ for all } y \in Y\}$. Hence if Q denotes the natural isometry of $Y^\perp$ onto $(X/Y)^*$, then $\delta(a) = Q(a \cdot \tilde{f} - \tilde{f} \cdot a)$, $a \in A$, defines a bounded derivation from A onto $(X/Y)^*$. Since A is amenable, there exists $h \in Y^\perp$ such that

$$\delta(a) = a \cdot Q(h) - Q(h) \cdot a \quad \text{for all } a \in A.$$

In particular if $g = \tilde{f} - h$, then $g \in Z(A, X^*)$ and g extends f .

(b) $\Rightarrow$ (c) Let $L = X^* \hat{\otimes} X$, the projective tensor product of X^* and X . Then L becomes a Banach A -bimodule with the action of A on L defined by

$$(f \otimes x) \cdot a = f \otimes x \cdot a, \quad a \cdot (f \otimes x) = f \otimes ax$$

for each $a \in A, x \in X$ and $f \in X^*$. Let H, K be the closed linear span of the

sets

$$\{T^*(f) \otimes x - f \otimes T(x); T \in \mathcal{C}, f \in X^* \text{ and } x \in X\},$$

$$\{f \otimes x; f \in Z(A, X^*) \text{ and } x \in X\}$$

respectively, where $\mathcal{C}$ denotes the family of all bounded linear operators from X into X commuting with the action of A on X . Let Y be the closed linear span of H and K . Then H, K are Banach A -submodules of L . Hence Y/H is also a Banach A -submodule of L/H . Let $\phi \in L^*$ be defined by $\langle \phi, f \otimes x \rangle = f(x)$ for all $f \in X^*$ and $x \in X$. Then $\phi \in H^\perp$. Hence we may define $\tilde{\Phi} \in (L/H)^*$ by $\tilde{\Phi}(y) = \phi(\bar{y})$ for all $y \in L$. Then, as readily checked, $\tilde{\Phi} \in Z(A, (Y/H)^*)$. So by assumption, there exists $\tilde{\Phi} \in Z(A, (L/H)^*)$ which extends Φ . Define

$$\langle P(f), x \rangle = \langle \tilde{\Phi}, \overline{f \otimes x} \rangle$$

for all $f \in X^*$ and $x \in X$. A routine computation shows that P is a bounded projection from X^* onto $Z(A, X^*)$ and P commutes with T^* for each $T \in \mathcal{C}$.

(c) $\Rightarrow$ (a) Let $\tilde{A}$ be A if A has a unit and the unitization of A otherwise. Let $\tilde{A}$ act on $X = \tilde{A} \hat{\otimes} \tilde{A}$ by

$$a \cdot (b \otimes c) = ab \otimes c \quad \text{and} \quad (b \otimes c) \cdot a = b \otimes ca$$

for all $a, b, c \in \tilde{A}$. Let $\mathcal{F} = \{L_a, R_a; a \in \tilde{A}\}$ be a family of bounded linear operators from X into X defined by

$$L_a(b \otimes c) = b \otimes ac \quad \text{and} \quad R_a(b \otimes c) = ba \otimes c$$

for all $a, b, c \in \tilde{A}$. Clearly, each operator in $\mathcal{F}$ commutes with the action of A on X . By assumption, there exists a bounded projection from X^* onto $Z(A, X^*) = Z(\tilde{A}, X^*)$ such that P commutes with T^* for each $T \in \mathcal{F}$. Let $q: X^* \rightarrow X^*$ be defined by

$$\langle q(f), a \otimes b \rangle = \langle f, b \otimes a \rangle$$

for each $f \in X^*, a, b \in \tilde{A}$. Let $M = q^*(P^*(e \otimes e))$ where e is the identity of $\tilde{A}$. We shall show that $M \in X^{**}$ is a virtual diagonal for $\tilde{A}$ i.e. $a \cdot M = M \cdot a$ and $(\pi^{**}M)a = a$ for all $a \in \tilde{A}$, where π is the bounded linear operator from $\tilde{A} \hat{\otimes} \tilde{A}$ to $\tilde{A}$ defined by $\pi(a \otimes b) = ab, a, b \in \tilde{A}$.

Indeed if $a \in \tilde{A}, \phi \in X^*$, then $q(\phi \cdot a) = L_a^*(q(\phi))$ and $q(a \cdot \phi) = R_a^*(q(\phi))$. Hence we have

$$\begin{aligned} \langle a \cdot M, \phi \rangle &= \langle M, \phi \cdot a \rangle = \langle P^*(e \otimes e), q(\phi \cdot a) \rangle \\ &= \langle P^*(e \otimes e), L_a^*(q(\phi)) \rangle = \langle e \otimes e, L_a^*P(q(\phi)) \rangle \\ &= \langle e \otimes a, P(q(\phi)) \rangle = \langle e \otimes e, a \cdot P(q(\phi)) \rangle. \end{aligned}$$

Similarly we have $\langle M \cdot a, \phi \rangle = \langle e \otimes e, P(q(\phi)) \cdot a \rangle$. Since $P(q(\phi)) \in Z(\tilde{A}, X^*)$, it follows that $a \cdot M = M \cdot a$.

Also since $q(\pi^*(\tilde{A}^*)) \subseteq Z(\tilde{A}, X^*)$, we have for each $f \in \tilde{A}^*$ and each $a \in \tilde{A}$

$$\begin{aligned}
\langle \pi^{**}M \cdot a, f \rangle &= \langle \pi^{**}M, a \cdot f \rangle = \langle M, \pi^*(a \cdot f) \rangle \\
&= \langle e \otimes e, P(q(\pi^*(a \cdot f))) \rangle = \langle e \otimes e, q(\pi^*(a \cdot f)) \rangle \\
&= \langle e \otimes e, \pi^*(a \cdot f) \rangle = \langle f, a \rangle.
\end{aligned}$$

Hence $\pi^{**}M \cdot a = a$. Consequently $\tilde{A}$ is amenable [7, Theorem 1.3]. In particular A is also amenable (see [6, p. 16]).

REMARK 2(a). Theorem 1 (a) $\Leftrightarrow$ (b) is due to J. Bunce [1, Proposition 2(a) $\Leftrightarrow$ (c)] (see also [3]) when A is a C^* -algebra with an identity. Also our condition (c) of the theorem implies condition (b) of Proposition 2 in [1].

(b). Let A be a C^* -algebra, and let $U(A_e)$ denote the group of unitary elements in the C^* -algebra A_e , where $A_e = A$ if A has an identity; otherwise A_e is the C^* -algebra formed by adjoining to A an identity. It then follows easily from [2, Lemma 1] and the proposition in [10] (see also [8]) that if A is strongly amenable and X is a Banach A -bimodule, then there exists a bounded projection from X^* onto $Z(A, X^*)$ such that

(i) $P(f) \in Z(A, X^*) \cap K(f)$ where $K(f)$ is the weak* closed convex hull of the set $\{u^*fu; u \in U(A_e)\}$;

(ii) P commutes with any weak*-continuous bounded linear operators from X^* into X^* commuting with the action of A on X^* .

(c). Property (b) of Theorem 1 is an analogue of the Silverman Hahn-Banach extension property for left amenable semigroups [9].

4. Ultraweakly amenable von Neumann algebras. Let M be a von Neumann algebra. A Banach M -bimodule X is *normal* if for each $\phi \in X^*$ and each $x \in X$, the functionals on M defined by $a \rightarrow \langle \phi, a \cdot x \rangle$, $a \rightarrow \langle \phi, x \cdot a \rangle$, $a \in M$, are ultraweakly continuous. We say that M is *ultraweakly amenable* if for any normal Banach M -bimodule X , every bounded derivation from M into X^* is inner (A. Connes [4] calls such von Neumann algebras amenable).

If H is a Hilbert space, let $\mathfrak{B}(H)$ denote the algebra of bounded linear operators from H into H .

THEOREM 3. *Let M be a von Neumann algebra. Then among the following conditions on M , we have (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Also if M_* is separable, then all the three conditions are equivalent.*

(a) M is ultraweakly amenable.

(b) For any normal Banach M -bimodule X and any Y Banach M -submodule of X , each linear functional in $Z(M, Y^*)$ has an extension to a linear functional in $Z(M, X^*)$.

(c) For any normal Banach M -bimodule X , there exists a bounded projection from X^* onto $Z(M, X^*)$ which commutes with any weak*-continuous bounded linear operator from X^* into X^* commuting with the action of M on X^* .

PROOF. The proof that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is similar to that of the corresponding implications in Theorem 1.

Now assume that M satisfies (c) and that M acts on a Hilbert space H such that the identity of M is the identity operator on H . Let $X = \mathfrak{B}(H)_*$, the

predual of $\mathfrak{B}(H)$, and consider X as a normal M -bimodule with the action on M on X defined by

$$\langle a \cdot \phi, t \rangle = \langle \phi, ta \rangle \quad \text{and} \quad \langle \phi \cdot a, t \rangle = \langle \phi, at \rangle$$

for all $\phi \in X$, $a \in A$ and $t \in \mathfrak{B}(H)$. Then by assumption, there exists a bounded projection P from $\mathfrak{B}(H)$ onto $Z(M, \mathfrak{B}(H)) = M'$, the commutant of M , and P commutes with any ultraweakly continuous bounded linear operators from $\mathfrak{B}(H)$ into $\mathfrak{B}(H)$ commuting with the action of M on $\mathfrak{B}(H)$. In particular $P(atb) = aP(t)b$ for any $a, b \in M'$ and $t \in \mathfrak{B}(H)$, i.e. P is a quasi-expectation of $\mathfrak{B}(H)$ on M' . By [11, Theorem 2], M' is injective. Hence M is injective (see [5, Theorems 5.1 and 5.3]). Consequently, if M_* is separable, then [4, Theorem 1] implies that (a) holds.

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REFERENCES

1. J. W. Bunce, *Characterizations of amenable and strongly amenable C^* -algebras*, Pacific J. Math. **43** (1972), 563–572.
2. ———, *Representations of strongly amenable C^* -algebras*, Proc. Amer. Math. Soc. **32** (1972), 241–246.
3. ———, *Finite operators and amenable C^* -algebras*, Proc. Amer. Math. Soc. **56** (1976), 145–151.
4. A. Connes, *On the cohomology of operator algebras* (preprint).
5. E. G. Effros and E. C. Lance, *Tensor products of operator algebras*, Advances in Math. **25** (1977), 1–34.
6. B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. No. 127, 1972.
7. ———, *Approximate diagonals and cohomology of certain annihilator Banach algebras*, Amer. J. Math. **94** (1972), 685–698.
8. A. T. Lau, *Semigroup of operators on dual Banach spaces*, Proc. Amer. Math. Soc. **54** (1976), 393–396.
9. R. J. Silverman, *Means on semigroups and the Hahn-Banach extension property*, Trans. Amer. Math. Soc. **83** (1956), 222–237.
10. F. J. Yeadon, *Fixed points and amenability: a counterexample*, J. Math. Anal. Appl. **45** (1974), 718–720.
11. J. W. Bunce and W. L. Paschke, *Quasi-expectations and amenable von Neumann algebras* (preprint).

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