PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 71, Number 1, August 1978

CONTRACTION SEMIGROUPS, STABILIZATION, AND THE MEAN ERGODIC THEOREM

ROBERT E. O'BRIEN, JR.

ABSTRACT. Consider the semidynamical system, $\dot{x} = Ax + Bu$, where A generates a C_0-contraction semigroup and B is bounded. If the system is controllable then it is weakly stabilizable. If in addition the semigroup is quasi-compact and B is compact, the mean ergodic theorem implies that the stability is uniform and exponential.

1. Introduction. A semidynamical system, $\dot{x} = Ax + Bu$ (A the generator of a C_0-contraction semigroup on a Hilbert space X, B a bounded operator from a Hilbert space Y to X), is stabilizable if for some bounded operator K from X to Y the system $\dot{x} = (A + BK)x$, has the zero solution asymptotically stable. The problem of stabilization of such a system was posed by M. Slemrod in **[7]**. Using a Lyapunov stability approach and the hypothesis that the semigroup generated by $A^* - BB^*$ was strongly almost periodic he showed that controllable systems were weakly-stabilizable.

In this note an equivalent condition to controllability is developed which (when coupled with an ergodic result of Foguel, **[3]**) removes the almost periodicity hypothesis in Slemrod's result. Subsequently, in IV.3, the Yosida-Kakutani mean ergodic theorem is used to show that if A generates a quasi-compact semigroup, the system is uniformly exponentially stabilizable. This generalizes Slemrod's result (Corollary 3.1 of **[7]**) to noncompact semigroups whose trajectories in $L(X, X)$ at some instant come "close" to a compact operator, e.g. a Markov process satisfying Doeblin's condition **[10]**.

This problem was suggested by C. T. Taam. These results owe much to his help and encouragement and form part of the author's dissertation presented to the George Washington University in December 1976.

II. Definitions and prerequisites.

II.1 Let X and Y be complex Hilbert spaces (with inner products, $\langle\, ,\, \rangle$ and $\langle\, ,\, \rangle_Y$), and $B \in L(Y, X)$ (the bounded linear operators from Y to X). We shall consider the semidynamical system on X,

$$\dot{x} = Ax + Bu, \qquad x(0) = 0, \tag{A,B}$$

where $u \in C$ (the Y-valued, strongly-measurable, locally L^2 functions on

Presented to the Society, August 25, 1976 under the title *Controllability, stabilization and the mean ergodic theorem*; received by the editors January 9, 1978.

AMS (MOS) subject classifications (1970). Primary 47D05; Secondary 47A50.

Key words and phrases. Contraction semigroup, controllability, stabilization.

$(-\infty, +\infty)$), and A is the infinitesimal generator of a C_0-semigroup, $T(t)$ $(t \geqslant 0)$, on X. A has domain $D(A)$ dense in X.

II.2 Define a solution of (A, B) to be an X-valued function, $x(t)$, defined and strongly continuous on $[0, +\infty)$, such that for every $y \in D(A^*)$, $\langle x(t), y\rangle$ is absolutely continuous on every interval, $[a, b]$, for $0 < a < b < +\infty$, and

$$\frac{d}{dt}\langle x(t), y\rangle = \langle x(t), A^* y\rangle + \langle Bu(t), y\rangle \quad \text{a.e.} \tag{2.1}$$

For each $u \in C$, a solution, $x(t)$, of (A, B) exists and is uniquely given by the Bochner integral

$$x(t) = \int_0^t T(t-s)Bu(s)\, ds. \tag{2.2}$$

II.3 Definition 1. The semidynamical system (A, B) is controllable if and only if $B^* T(t)^* x = 0$ for all $t \geqslant 0$ implies $x = 0$.

This definition is equivalent to the customary notion of controllability for the system (A, B), i.e. the subspace,

$$\mathrm{sp}\left\{ \int_0^t T(t-s)Bu(s)\, ds \colon t > 0, u \in C \right\}$$

is dense in X. (See [**1**], [**2**], or [**7**].)

II.4 Fundamental to our discussion is the perturbation result of R. S. Phillips, [**6**]: if A generates a C_0-semigroup, $T(t)$ $(t \geqslant 0)$, on a Banach space X, and $Q \in L(X, X)$, then $A + Q$ (with $D(A+Q)$ defined to be $D(A)$) also generates a C_0-semigroup, $V(t)$ $(t \geqslant 0)$, and

$$V(t) = T(t) + \int_0^t T(t-s)QV(s)\, ds. \tag{2.3}$$

(The author is indebted to R. Datko for pointing out that the original proofs of Lemmas 1 and 2 could be simplified using 2.3.)

Definition 2. Let $K \in L(X, Y)$, (A, B) be the semidynamical system defined in II.1, and let $V(t)$ $(t \geqslant 0)$ be the C_0-semigroup generated by $A + BK$. Then K *weakly* (*strongly*, *uniformly*) *stabilizes* (A, B) if and only if $\lim_{t\to\infty} V(t) = 0$ in the weak (strong, uniform) operator topology. (These topologies are defined in [**9**, p. 111], and are taken in the usual sense.)

II.5 A digression on notation; the weak (strong) limit on X will be written w-lim (s-lim), while the weak (strong) closure of a set S will be denoted S^w (S^a). Also, $O(x)$ will refer to the orbit, $\{T(t)x \colon t \geqslant 0\}$.

Definition 3. The C_0-semigroup $T(t)$ $(t \geqslant 0)$ is strongly almost periodic if and only if $O(x)^a$ is strongly compact for all $x \in X$.

II.6 The proofs of Theorems 1 and 2 will require the collection (from diverse papers) of some pertinent facts about a C_0-contraction semigroup, $T(t)$ $(t \geqslant 0)$, generated by A, on X.

We shall need the theorem of S. Foguel, [**3**], characterizing the isometric subspace of $T(t)$ and $T(t)^*$ $(t \geqslant 0)$ defined by

$$X_u = \{x: \|T(t)x\| = \|x\| = \|T(t)^*x\|, t > 0\}.$$

THEOREM. *Let* $T(t)$ $(t \geqslant 0)$ *be a* C_0*-contraction semigroup on the Hilbert space* X, *with isometric subspace* X_u. *Then* X_u *is a closed* $T(t)$, $T(t)^*$ $(t \geqslant 0)$ *invariant subspace*; $T(t)$ $(t \geqslant 0)$ *forms a* C_0*-semigroup of unitary operators on* X_u; *and for* y *orthogonal to* X_u

$$\operatorname*{w-lim}_{t\to\infty} T(t)y = \operatorname*{w-lim}_{t\to\infty} T(t)^*y = 0.$$

In view of this result, X_u will be called the unitary subspace of $T(t)$ $(t \geqslant 0)$. (X_u may be $\{0\}$, in which case we say that $T(t)$ $(t > 0)$ is completely nonunitary.)

DEFINITION 4. A C_0-semigroup, $T(t)$ $(t \geqslant 0)$, on a Banach space X is *quasi-compact* if and only if for some compact operator, K, in $L(X, X)$ and some $p > 0$

$$\|T(p) - K\| < 1.$$

The formulation of the Yosida-Kakutani Mean Ergodic Theorem, **[10]** stated here is due to C. T. Taam; his proof (based on the Glicksberg and deLeeuw compactification of **[4]**) was given in **[8]**, and appears in the appendices of **[5]**.

THEOREM. *Let* $T(t)$ $(t \geqslant 0)$ *be a uniformly bounded, quasi-compact* C_0*-semigroup on a Banach space* X. *Then*

(a) $T(t)$ $(t \geqslant 0)$ *is strongly almost periodic.*

(b) *There exists a compact projection,* P, *in the uniform closure of* $T(t)$ $(t \geqslant 0)$, *real numbers* $a_1, \ldots, a_m$, *and projections* $P_1, \ldots, P_m$ *in* $L(X, X)$ *such that* $P_iP_k = 0$ *for* $i \neq k$, $P = P_1 + \cdots + P_m$, *and*

$$T(t)P_n = e^{ia_nt}P_n \qquad (t \geqslant 0).$$

(c) *There exist* $a > 0$ *and* $M \geqslant 1$ *such that*

$$\|T(t)(I - P)\| \leqslant Me^{-at} \qquad (t \geqslant 0).$$

III. Two lemmas and a corollary.

III.1 LEMMA 1. *Let* $T(t)$ $(t \geqslant 0)$ *be a* C_0*-semigroup on a Banach space* X, $Q \in L(X, X)$, *and* A *the generator of* $T(t)$ $(t \geqslant 0)$. *Let* $U(t)$ $(t \geqslant 0)$ *be the* C_0*-semigroup generated by* $A + Q$. *Then for all* $x \in X$ *the following are equivalent*:

1. $QU(t)x = 0$ *for all* $t \geqslant 0$.
2. $U(t)x = T(t)x$ *for all* $t \geqslant 0$.
3. $QT(t)x = 0$ *for all* $t \geqslant 0$.

PROOF. Assume $QU(t)x = 0$ $(t \geqslant 0)$, then from 2.3

$$U(t)x = T(t)x + \int_0^t T(t - s)QU(s)x\,ds = T(t)x \qquad (t \geqslant 0).$$

Moreover, if $U(t)x = T(t)x$ for $t \geqslant 0$, (again from 2.3)

$$\int_0^t T(t-s)QU(s)x\,ds = U(t)x - T(t)x = 0 \qquad (t \geqslant 0). \tag{3.1}$$

Now for $z \in D(A^*)$, $t \to \int_0^t \langle QU(s)x, T(t-s)^* z\rangle\,ds$ is differentiable and

$$\frac{d}{dt}\int_0^t \langle QU(s)x, T(t-s)^* z\rangle\,ds$$

$$= \langle QU(t)x, z\rangle + \left\langle \int_0^t T(t-s)QU(s)x\,ds, A^* z\right\rangle. \tag{3.2}$$

Hence 3.1, 3.2, and the density of $D(A^*)$ imply that $QU(t)x = 0$ for all $t \geqslant 0$, and we have shown that 1 is equivalent to 2. Similarly 3 is equivalent to 2. Q.E.D.

III.2 Corollary. *Let X and Y be Hilbert spaces, $B \in L(Y, X)$, $T(t)$ $(t \geqslant 0)$ the C_0-semigroup with generator A and $S(t)$ $(t \geqslant 0)$ the C_0-semigroup generated by $A - BB^*$. The following are equivalent*:

1. *The system (A, B) is controllable.*
2. *For $x \in X$, $T(t)^* x = S(t)^* x$ for all $t \geqslant 0$ implies $x = 0$.*
3. *The system $(A - BB^*, B)$ is controllable.*

III.3 Lemma 2. *Let $T(t)$ $(t \geqslant 0)$ and $S(t)$ $(t \geqslant 0)$ be C_0-contraction semigroups on the Hilbert space X. If $Q \in L(X, X)$ is compact, then for each $t \geqslant 0$ the integral*

$$\int_0^t T(t-s)QS(s)\,ds$$

defines a compact operator on X.

Proof. Take $S_1 = \{x: \|x\| \leqslant 1\}$. Then the subspace spanned by the compact set $Q(S_1)^a$ is separable and hence has a countable complete orthonormal base, $\{z_n\}$. For each n, $n = 1, 2, \ldots,$ define $P_n \in L(X, X)$ by

$$P_n x = \sum_{m=1}^{n} \langle x, z_m\rangle z_m, \qquad x \in X.$$

An application of Dini's theorem to the sequence of continuous functions, $x \to \|P_n x - x\|$ (which are defined and monotonically decreasing to 0 on the compact set $Q(S_1)^a$) gives

$$\lim_{n\to\infty}\left[\sup\{\|P_n y - y\| : y \in Q(S_1)^a\}\right] = 0.$$

Consequently $\lim_{n\to\infty}\|P_n Q - Q\| = 0$ and hence

$$\lim_{n\to\infty}\|T(t-s)P_n QS(s) - T(t-s)QS(s)\| = 0 \quad \text{uniformly on } 0 \leqslant s \leqslant t.$$

But

$$T(t-s)P_n QS(s)x = \sum_{m=1}^{n} \langle x, S(s)^* Q^* z_m\rangle T(t-s)z_m, \qquad \|x\| \leqslant 1,$$

and therefore, for each fixed n, $n = 1, 2, \ldots, s \to T(t-s)P_n QS(s)$ is uniform operator continuous on $0 \leqslant s \leqslant t$. Hence $s \to T(t-s)QS(s)$ is uniform operator continuous on $0 \leqslant s \leqslant t$, and therefore the integral, $\int_0^t T(t$

$-s)QS(s)\,ds$ exists and is compact, being the uniform operator limit of compact Riemann sums,

$$\sum_{k=1}^{n} \Delta s_k T(t-s_k')QS(s_k').$$

IV. The main results.

IV.1 THEOREM 1. *Let X and Y be Hilbert spaces, $B \in L(Y, X)$, and let $T(t)$ $(t \geqslant 0)$ be a C_0-contraction semigroup on X with generator A. If the system (A, B) is controllable, then $-B^*$ weakly stabilizes the systems (A, B) and (A^*, B).*

PROOF. Since $T(t)$ $(t \geqslant 0)$ is a C_0-contraction semigroup, A and hence A^* and $A - BB^*$ are dissipative. Therefore, the C_0-semigroups $S(t)$ and $S(t)^*$ $(t \geqslant 0)$ generated by $A - BB^*$ and $A^* - BB^*$ respectively consist of contractions.

Consider the unitary subspace of $S(t)$ $(t \geqslant 0)$, X_u, defined in II.6. Since X_u is $S(t)^*$ $(t \geqslant 0)$ invariant (by Foguel's Theorem, II.6), $X_u \cap D(A^* - BB^*)$ is dense in X_u. Take $x \in X_u \cap D(A^* - BB^*)$. Then since $S(t)^*x \in D(A^*)$ $(t \geqslant 0)$, while $\|S(t)^*x\|^2 = \|x\|^2$ $(t \geqslant 0)$ and A^* is dissipative, for $t \geqslant 0$

$$\frac{d}{dt}\|S(t)^*x\|^2 = 0 \tag{4.1}$$

and

$$\frac{d}{dt}\|S(t)^*x\|^2 = 2\mathrm{Re}\langle (A^* - BB^*)S(t)^*x, S(t)^*x\rangle \leqslant -\|B^*S(t)^*x\|^2. \tag{4.2}$$

Hence $B^*S(t)^*x = 0$ for all $t \geqslant 0$.

By the corollary of III.2, if the system (A, B) is controllable then the system $(A - BB^*, B)$ is controllable, and hence 4.3 implies that $x = 0$. Since x was arbitrary and $X_u \cap D(A^* - BB^*)$ is dense in X_u, X_u must be the null subspace, $\{0\}$. But then from Foguel's theorem of II.6, for all $x \in X$,

$$\underset{t\to\infty}{\text{w-lim}}\, S(t)x = \underset{t\to\infty}{\text{w-lim}}\, S(t)^*x = 0, \quad \text{i.e.} \quad \lim_{t\to\infty} S(t) = \lim_{t\to\infty} S(t)^* = 0$$

in the weak operator topology. Q.E.D.

IV.2 COROLLARY. *Let X and Y be Hilbert spaces, $B \in L(Y, X)$, A and $A - BB^*$ the generators of C_0-contraction semigroups $T(t)$ $(t \geqslant 0)$ and $S(t)$ $(t \geqslant 0)$ respectively. If the system (A, B) is controllable then $S(t)$ $(t \geqslant 0)$ is completely nonunitary.*

IV.3 THEOREM 2. *Let X and Y be Hilbert spaces, $B \in L(Y, X)$, and let A be the generator of a quasi-compact C_0-contraction semigroup, $T(t)$ $(t \geqslant 0)$, on X, and take $S(t)$ $(t \geqslant 0)$ to be the C_0-semigroup generated by $A - BB^*$. If B is compact and the system (A, B) is controllable, then $-B^*$ uniformly stabilizes the systems (A, B) and (A^*, B); moreover, there exist real numbers $a > 0$ and $M \geqslant 1$ with $\|S(t)^*\| = \|S(t)\| \leqslant Me^{-at}$ $(t \geqslant 0)$.*

PROOF. Since $T(t)$ $(t \geqslant 0)$ is assumed quasi-compact, $\|T(p) - K\| < 1$ for some compact $K \in L(X, X)$ and some $p > 0$. But then (from 2.7)

$$\left\| S(p) - \left[K - \int_0^p T(p-s) BB^* S(s)\, ds \right] \right\| = \|T(p) - K\| < 1$$

and since B is assumed compact, $\int_0^p T(p-s)BB^*S(s)\,ds$ is compact (Lemma 2) and therefore $S(t)$ $(t \geqslant 0)$ is quasi-compact. Hence, $S(t)$ $(t \geqslant 0)$ being uniformly bounded and quasi-compact, we may apply the mean ergodic theorem (II.7) to get the orthogonal projection P, and real numbers, $a > 0$ and $M \geqslant 1$ with $\|S(t)(I-P)\| \leqslant Me^{-at}$ for all $t \geqslant 0$. This implies that $PX = X_p$ coincides with X_u, the unitary subspace of $S(t)$ $(t \geqslant 0)$.

The assumed controllability of the system (A, B) and the argument of IV.1 require that $X_u = PX$ be $\{0\}$ however, and hence $P = 0$ and $\|S(t)\| \leqslant Me^{-at}$ $(t \geqslant 0)$. Noting that $\|S(t)\| = \|S(t)^*\|$ we are finished.

IV.4 The duality here is intrinsic. The conclusions of these results hold if the system (A, B) is replaced by its adjoint, (A^*, B).

REFERENCES

1. A. V. Balakrishnan, *Introduction to optimization theory in Hilbert space*, Springer, Berlin, 1973.

2. H. O. Fattorini, *On complete controllability of linear systems*, J. Differential Equations **3** (1967), 391–402.

3. S. R. Foguel, *The ergodic theorem for Markov-processes*, Israel J. Math. **4** (1966), 11–22.

4. I. Glicksberg and K. deLeeuw, *Applications of almost periodic compactifications*, Acta. Math. **105** (1961), 63–97.

5. R. E. O'Brien, *Controllability, stabilization and contraction semigroups in Hilbert space*, Dissertation, George Washington Univ., 1977.

6. R. S. Phillips, *Perturbation theory for semigroups of linear operators*, Trans. Amer. Math. Soc. **74** (1953), 199–221.

7. M. Slemrod, *A note on complete controllability and stabilizability of linear control systems in Hilbert space*, SIAM J. Control and Optimization **12** (1974), 500–508.

8. C. T. Taam, *Lectures on differential equations on a Banach space* (unpublished).

9. K. Yosida, *Functional analysis*, Springer-Verlag, Berlin, 1966.

10. K. Yosida and S. Kakutani, *Operator theoretical treatment of Markov processes and mean ergodic theorems*, Ann. of Math. (2) **42** (1941), 188–228.

DELTA PROJECT, GODDARD SPACE FLIGHT CENTER, GREENBELT, MARYLAND 20795

DEPARTMENT OF MATHEMATICS, GEORGE WASHINGTON UNIVERSITY, WASHINGTON, D.C. 20052

Current address: Aerospace Systems Section, Computer Science Corporation, Falls Church, Virginia 22041