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ON THE FUNDAMENTAL GROUP OF A COMPACT NEGATIVELY CURVED MANIFOLD

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ABSTRACT. A question of Hirsch and Thurston on the fundamental group of a compact negatively curved manifold is investigated.

1. In [8], Hirsch and Thurston have asked the question whether the fundamental group $\pi_1(M)$ of a compact Riemannian manifold M of negative sectional curvature is in the class C of groups which contains all amenable groups and which satisfies: if G and H are in C, then the free product G^*H is in C, and if G has finite index in K then K is in C. This question is related to their investigation [8] on the Euler characteristic $\chi(M)$ of M. In this paper, we shall give some evidence for $\pi_1(M)$ of a compact Riemannian manifold M of negative sectional curvature not in this class C of groups. According to [8], $\pi_1(M)$ in C would imply that the Euler characteristic $\chi(M)$ vanishes. Consequently, our result to a certain extent supports the truth of a well-known conjecture that for a compact negatively curved manifold M of even dimension n, $\chi(M) < 0$ for $n = 2 \bmod 4$ and $\chi(M) \geqslant 0$ for $n = 0 \bmod 4$.

We first show that any amenable subgroup of $\pi_1(M)$ is infinite cyclic. If M has nonpositive sectional curvature, then it is not known whether any amenable subgroup of $\pi_1(M)$ is a finite extension of abelian groups. When the subgroup is solvable, this is known to be true [7], [12].

Next, we show the following statements:

(1) $\pi_1(M)$ has relations for a certain set of generators.[1]

(2) A subgroup of $\pi_1(M)$ with an amenable subgroup of axial elements of finite index is an amenable group of axial elements and is infinite cyclic.

(3) A free product of amenable subgroups of axial elements is free.

(4) A subgroup of $\pi_1(M)$ with a free product subgroup of amenable subgroups of axial elements of finite index is free.

(5) If $\pi_1(M)$ belongs to the class C, then $\pi_1(M)$ is free. We shall use a deep result of Stallings [13] and Swan [14] to prove statement (4). Stallings has proved that a finitely generated torsion free group with a free subgroup of finite index is free [13]. In [14], Swan has extended this theorem to infinitely generated groups. A direct proof of statement (4) seems to be possible.

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[1]This statement is not enough to imply that $\pi_1(M)$ is not a free group.

2. Let M be a compact negatively curved manifold. Then the sectional curvature is bounded from above by a negative constant. The group G of covering transformations associated to the universal covering $\tilde{M}$ of M is isomorphic to $\pi_1(M)$. G is the disjoint union of its stability groups G_x, $x \in \tilde{M}(\infty)$ [5]. Here $\tilde{M}(\infty)$ denotes the boundary of $\tilde{M}$ [5]. All elements of G are axial and they translate certain geodesics [3]. There are exactly two fixed points in $\tilde{M}(\infty)$ of each axial elements. Each G_x is infinite cyclic. The limit set $L(G)$ of G is $\tilde{M}(\infty)$.

THEOREM 1. *Let M be a compact negatively curved manifold. Any amenable subgroup of $\pi_1(M)$ is infinite cyclic.*

PROOF. Let H be any subgroup of G. The limit set $L(H)$ of H is one of the following [5]: (1) one point and every element of H is parabolic, (2) two points and H is infinite cyclic of axial elements, (3) a perfect nowhere dense set in $\tilde{M}(\infty)$ and (4) $\tilde{M}(\infty)$. It is clear that (1) cannot occur. If (3) and (4) occur, then we may use Eberlein's freeness argument [4] to show that there is a free subgroup F of at least two generators of H. Hence H cannot be amenable. Thus H is infinite cyclic and consists of axial elements of the same fixed points.

COROLLARY. *Every abelian or solvable subgroup of $\pi_1(M)$ is infinite cyclic* (*see* [1], [11]).

THEOREM 2. *Let M be a compact negatively curved manifold. There is at least one relation for a certain set of generators of $\pi_1(M)$.*

PROOF. Let G be a discrete group of isometries of $\tilde{M}$. One can form the Dirichlet fundamental region $\mathfrak{R}$ for G in $\tilde{M}$. $\mathfrak{R} = \{p \in \tilde{M} | d(p, p_0) < d(p, \phi(p_0)), \forall\phi \in G \text{ and } \phi \neq \text{id}\}$. Here p_0 is a point not fixed by G. Then the following properties are known [6]:

(1) Every point in $\tilde{M}$ is identified with exactly one point of $\mathfrak{R}$ or finitely many points of the boundary Bd ($\mathfrak{R}$).

(2) $\mathfrak{R}$ is not empty and $\text{Bd}(\mathfrak{R}) = \{p \in \tilde{M} | d(p, p_0) \leqslant d(p, p_j), \forall j > 0$ and $d(p, p_0) = d(p, p_i)$ for some $i > 0\}$, where $G = \{\phi_i\}$ and $p_i = \phi_i(p_0)$, $i > 0$.

(3) Each compact subset K of $\tilde{M}$ meets only finitely many images $\phi(\overline{\mathfrak{R}})$, $\phi \in G$ of $\mathfrak{R}$ under G. The set $S_i = \overline{\mathfrak{R}} \cap \{p \in \tilde{M} | d(p, p_0) = d(p, p_i)\}$ is called the side of $\mathfrak{R}$ determined by ϕ_i. If S_i is the side of $\mathfrak{R}$ determined by ϕ_i and S_i' is the side of $\mathfrak{R}$ determined by ϕ_i^{-1}, then S_i is conjugate to exactly one other side S_i' of $\mathfrak{R}$.

Since M is compact, the Dirichlet region $\mathfrak{R}$ for G in $\tilde{M}$ is compact and consists of finitely many sides which intersect at lower dimensional faces. Let T be a lower dimensional face in Bd ($\mathfrak{R}$) which lies on some side S_{k_1} determined by $\phi_{k_1} \in G$. Then S_{k_1} is conjugate to S_{k_1}'. $\phi_{k_1}(T)$ lies on S_{k_1}' and some other side S_{k_2} determined by ϕ_{k_2}. Then $\phi_{k_2}\phi_{k_1}(T)$ lies on S_{k_2}' and some other side. Consider an orbit of T under these conjugations. Then this orbit must be finite and is $\{T, \phi_{k_1}(T), \ldots, \phi_{k_t}\phi_{k_{t-1}} \cdots \phi_{k_1}(T) = T\}$. Since $\phi_{k_t}\phi_{k_{t-1}} \cdots \phi_{k_1}$ is axial and leaves a proper subset T of M invariant, it has to be the identity. Thus G has a relation.

3. We need a theorem due to Swan [**14**] which generalizes a theorem of Stallings [**13**].

THEOREM (SWAN). *A torsion free group (finitely or infinitely generated) with a free subgroup F of finite index is a free group.*

PROPOSITION 1. *A subgroup of $\pi_1(M)$ with an amenable subgroup of axial elements of finite index is an amenable subgroup of axial elements and is infinite cyclic.*

PROOF. Since the limit set $L(H)$ of H with an amenable subgroup G_x $(x \in \tilde{M}(\infty))$ of axial elements of finite index is equal to the limit set $L(G_x)$, $L(H)$ consists of two points and is an infinite cyclic group of axial elements.

PROPOSITION 2. *A free product of amenable subgroups of axial elements is a free group.*

PROOF. Each amenable group of axial elements is infinite cyclic. The given free product is free from the definition of free products.

PROPOSITION 3. *A subgroup of $\pi_1(M)$ with a free product subgroup of amenable subgroups of axial elements of finite index is a free group.*

PROOF. Since $\pi_1(M)$ is torsion free, the given group is also. We may apply Swan's theorem to obtain the statement.

REMARK. Can one give a geometric proof of the above proposition?

THEOREM 3. *Let M be a compact negatively curved manifold. If $\pi_1(M)$ belongs to the class C of groups then $\pi_1(M)$ is free.*

PROOF. Each group in C is obtained by a composition of operations described in Propositions 1, 2 and 3. The resulting group is a free group.

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