AN EXTREMAL PROBLEM FOR QUASICONFORMAL MAPPINGS IN AN ANNULUS¹

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ABSTRACT. The following extremal problem is solved. We consider a family of continuously differentiable univalent quasiconformal mappings $w = f(z)$ of the annulus $r < |z| < 1$ onto the unit disk minus some continuum containing the origin. For a point b on a fixed circle, maximize $|f(b)|$ within the family.

The problem is solved by using a variational method due to Schiffer. The extremal function and the maximum are found in terms of the Weierstrass $\wp$ -function and the elliptic modular function.

In his investigation of numerical construction of conformal mappings, D. Gaier [3] considered the family F of functions $f(z)$ regular, analytic and univalent in the annulus $R: r < |z| < 1$ and satisfying the following three conditions:

$$|f(z)| < 1 \quad \text{in } R \text{ while } |f(z)| = 1 \quad \text{on } |z| = 1;$$

$$\text{that is, } f(z) \text{ maps } R \text{ onto the unit disk minus some continuum } \Gamma; \quad (1)$$

$$f(z) \neq 0 \quad \text{in } R, (\Gamma \text{ contains the origin}); \quad (2)$$

$$f(1) = 1, \text{ (a normalization).} \quad (3)$$

Gaier raised the question of finding the maximum of $|f(z) - z|$ for all $f \in F$ and $z \in R$. Duren and Schiffer [2] solve this problem by a method of variation within the family F . (See also Gehring and Hällström [5] and Gaier and Huckemann [4].)

In order to answer the question raised by Gaier, Duren and Schiffer first solve the extremal problem:

$$\max |f(b)|, \quad f \in F, \quad \text{where } |b| = r_1, \quad r < r_1 < 1. \quad (*)$$

This problem is of independent interest. Grötzsch [6] previously solved it by the method of extremal length.

In the present paper the requirement of analyticity is weakened and problem (*) is solved within a family F' of quasiconformal mappings by a variational method.

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1. Let $w = f(z) = u(x, y) + iv(x, y)$ be a family of continuously differentiable univalent mappings of a domain D_z in the z plane onto a domain D_w in the w plane.

We call the mapping K -quasiconformal in D_z if it satisfies the inequality²

$$|f_z|^2 + |f_{\bar{z}}|^2 \leq K(|f_z|^2 - |f_{\bar{z}}|^2); \quad K \geq 1, \quad (4)$$

$$\left| \frac{f_{\bar{z}}}{f_z} \right|^2 \leq \frac{K-1}{K+1} \quad (4')$$

for all points of D_z . Note that

$$|f_z|^2 + |f_{\bar{z}}|^2 = \frac{1}{2}(u_x^2 + u_y^2 + v_x^2 + v_y^2) \quad (5)$$

and

$$|f_z|^2 - |f_{\bar{z}}|^2 = u_x v_y - u_y v_x = \partial(u, v) / \partial(x, y) \quad (6)$$

is the Jacobian of the mapping [1].

Denote the family $f(z)$ of univalent, class C^1 functions satisfying (1), (2), (3), (4) as F' . Consider the extremal problem:

$$\max |f(b)|, \quad f \in F' \quad \text{where } |b| = r_1, \quad r < r_1 < 1. \quad (7)$$

Problem (7) is solved by a variational method. In the variational method we seek that function (if it exists) in the family for which the functional $\chi[f] = |f(b)|$ attains its maximum value. The value of the functional is compared for functions of the family which are near each other in a certain sense. A comparison function within the family F' is constructed by a method of Schiffer [12], [2]. The existence of the extremal function can be shown as in [12], [10].

Let $w = f(z)$ be a given function in F' and let Γ denote the complement within the open disk of the range $\Delta(f)$ of f . Let $w_0 \neq 0$ be a fixed point on the continuum Γ . Corresponding to each point w_0 it is known [2] that there exist variation functions $V_\rho(w)$ analytic and univalent in the range of f having the following properties:

- (i) $\lim_{\rho \rightarrow 0} V_\rho(w) = w$;
- (ii) $|V_\rho(w)| = 1$ for $|w| = 1$;
- (iii) $V_\rho(w) \neq 0$ for w in $\Delta(f)$;
- (iv) $V_\rho(1) = 1$.

Such a function is

$$V(w) = w \left[1 + \frac{a\rho^2(1-w)}{(w-w_0)(1-\bar{w}_0 w_0)} + \frac{\bar{a}\rho^2(1-w)}{(1-\bar{w}_0 w)(1-\bar{w}_0 \bar{w}_0)} + O(\rho^3) \right] \quad (8)$$

for proper choice of the coefficient a . A comparison function in the class F' is

$$f^*(z) = V_\rho[f(z)]. \quad (9)$$

²This is not the currently customary definition.

Suppose $f(z)$ is the extremal function, and denote $f(b) = B$. We have $|V_\rho[f(b)]| \leq |f(b)|$. Introducing (8) this yields

$$\left| 1 + \frac{a\rho^2(1-B)}{(B-w_0)(1-w_0)w_0} + \frac{\bar{a}\rho^2(1-B)}{(1-\bar{w}_0B)(1-\bar{w}_0)\bar{w}_0} + O(\rho^3) \right| \leq 1. \quad (10)$$

Noting that $\operatorname{Re}\{\alpha\} = \operatorname{Re}\{\bar{\alpha}\}$, and $|B| < 1$, a calculation yields

$$\operatorname{Re}\left\{ \frac{a\rho^2}{w_0(B-w_0)(1-w_0\bar{B})} + O(\rho^3) \right\} \leq 0. \quad (11)$$

2. From the fundamental lemma of boundary variation [11] in the theory of conformal mapping, we can conclude from inequality (11), which is valid for all admissible variations, that Γ is an analytic arc $w = w(t)$ satisfying the differential equation

$$\frac{(w')^2}{w(B-w)(1-w\bar{B})} > 0. \quad (12)$$

The analytic arc Γ must pass through the origin since $f(z) \neq 0$ in R .

Obviously the ray $w = Bt$, $0 \leq t \leq 1$, satisfies (12) and contains the origin. Because of the uniqueness theorem for the differential equation [2], this segment is the only solution containing the origin and hence Γ must be a part of it.

Thus we prove that the extremal function f maps the annulus R onto the unit disk slit along a segment starting at the origin in the direction of B .

3. Without loss of generality we can assume that $B > 0$. Indeed, let $B = |B|e^{i\beta}$ and let α be defined by $f(e^{i\alpha}) = e^{i\beta}$. Then the function $\varphi(z) = e^{-i\beta}f(e^{i\alpha}z)$ also belongs to F' and maps R onto the disk slit along the real axis and takes the value $|B|$ on the circle $|z| = |b|$.

4. In order to characterize the extremal function $f(z)$ another variation will be introduced. We choose a point w_0 in the range of f and define a variation $\tilde{V}_\rho(w)$ which has the form (8) for $|w - w_0| \geq \rho$, but must be modified inside the disk to remain of class C^1 .

$$\begin{aligned} \tilde{V}_\rho(w) = w \left[1 + \frac{a(1-w)}{w_0(1-w_0)} \cdot \overline{(w-w_0)} \left(2 - \frac{|w-w_0|^2}{\rho^2} \right) \right. \\ \left. + \frac{\bar{a}\rho^2(1-w)}{(1-\bar{w}_0w)(1-\bar{w}_0)\bar{w}_0} + O(\rho^3) \right] \quad (13) \end{aligned}$$

satisfies the requirement. An easy calculation shows that our functional $\chi[f] = |f(b)|$ satisfies the asymptotic equation

$$\chi[f^*] = \chi[f] \left[1 + \operatorname{Re} \frac{a\rho^2(1 - B^2)}{w_0(B - w_0)(1 - Bw_0)} + O(\rho^3) \right] \quad (14)$$

since we may assume that for sufficiently small ρ , B is outside the disk $|w - w_0| \leq \rho$.

The assumption that $f(z)$ is the extremal function for our problem implies that $\chi[f^*] \leq \chi[f]$ for all variations $\tilde{V}_\rho[w]$ which preserve the K -quasiconformality of the varied function. This condition allows the application of [12]. It is seen that the extremal function $w = f(z)$ satisfies the relation

$$J(w) - \sqrt{K - 1/K + 1} \overline{J(w)} = k(z) \quad (15)$$

where

$$J(w) = \int_1^w \frac{dw}{\sqrt{w(B - w)(1 - wB)}}, \quad w = f(z) \quad (16)$$

and $k(z)$ is analytic. Note that $k(z)$ is multivalued and has a singularity at b .

The extension (13) of (8) into $|w - w_0| < \rho$ may lack a uniformity property that is needed in [12] as was pointed out by Renelt [10]. This is remedied by using the solution of a Beltrami equation as shown in [14]. The rest of the procedure remains unchanged.

We can transform (15) so as to express $J(w)$ in terms of $k(z)$ and $\overline{k(z)}$.

$$J(w) = \frac{K+1}{2} k(z) + \frac{1}{2} \sqrt{K^2 - 1} \overline{k(z)}. \quad (15')$$

Differentiating (15') with respect to x yields

$$\frac{1}{\sqrt{w(B - w)(1 - Bw)}} f_x = \frac{K+1}{2} k'(z) + \frac{1}{2} \sqrt{K^2 - 1} \overline{k'(z)}. \quad (15'')$$

Let us consider that point on $|z| = r$ which corresponds to the right endpoint of the slit Γ . Obviously, at that point the Jacobian of the mapping vanishes and hence $f_x = 0$. This means that $k'(z)$ has a zero on the circle $|z| = r$.

5. In order to study the boundary behavior of the analytic function $k(z)$, we observe that the circumference $|z| = 1$ is mapped by $f(z)$ onto $|w| = 1$. On $|w| = 1$ we can write

$$dJ(w) = \frac{dw}{\sqrt{w(B - w)(1 - Bw)}} = \frac{ie^{i\alpha} d\alpha}{\sqrt{e^{i\alpha}(B - e^{i\alpha})(1 - Be^{i\alpha})}}$$

where $w = e^{i\alpha}$. Hence

$$dJ(w) = \frac{id\alpha}{\sqrt{(B - e^{i\alpha})(e^{-i\alpha} - B)}} = \frac{d\alpha}{|B - e^{i\alpha}|} = \text{real}.$$

In view of (15) we can assert that $dk(z)$ is real for $|z| = 1$.

Next consider the image of $|z| = r$ which we have shown to be a segment of the positive real axis. Since this segment does not reach the point B which

corresponds to b ($r < |b| < 1$), $J(w)$ is real on that segment and so is $k(z)$ in view of (15).

Since we can parametrize $z = e^{i\alpha}$ and $z = re^{i\alpha}$ on the two circumferences, we have in both cases $dz = iz d\alpha$ and from the observation that $k'(z)dz =$ real in each case, we infer that on each circumference

$$l(z) = z^2 k'(z)^2 \leq 0. \quad (17)$$

From the definition of $k(z)$ it is easily seen that $k'(z)$ is unbounded in the neighborhood of $z = b$, while $\sqrt{z - b} k'(z)$ is regular.

Since $l(z)$ is real for $|z| = r$ and $|z| = 1$, by the Schwarz reflection principle, we have

$$l(1/\bar{z}) = \overline{l(z)} = l(r^2/\bar{z}).$$

Hence $l(r^2z) = l(z)$. Now, let $t = \log z$, $L(t) = l(z)$. Then, in the rectangle

$$\log r \leq \operatorname{Re}\{t\} < \log 1/r, \quad 0 \leq \operatorname{Im}\{t\} < 2\pi$$

the function $L(t)$ is regular analytic except for two simple poles at $t_1 = \log b$ and $t_2 = \log 1/\bar{b}$. It is doubly periodic

$$L(t + 2 \log r) = L(t), \quad L(t + 2\pi i) = L(t).$$

In view of (17), $L(t)$ is negative for $\operatorname{Re}\{t\} = 0$ and $\operatorname{Re} t = \log r$. Consider the half period parallelogram

$$\log r < \operatorname{Re}\{t\} \leq 0, \quad \gamma \leq \operatorname{Im} t \leq \gamma + \pi$$

where $b = |b|e^{i\gamma}$. We can map this parallelogram onto the upper half-plane such that the point $\log b$ on its boundary goes into infinity. The mapping function $\psi(t)$ is easily seen to be doubly periodic with the same periods as $L(t)$ and with the same poles. Hence, by the well-known uniqueness theorem for elliptic functions

$$L(t) = A\psi(t) + C.$$

Since $\psi(t)$ and $L(t)$ are both real for $\operatorname{Re}\{t\} = 0$ and $\operatorname{Re}\{t\} = \log r$, clearly A and C are real constants. We also see that $L(t)$ gives a one-to-one map of the boundary of the half period parallelogram onto the real axis. Hence, it has precisely one simple zero which corresponds to the zero of $k'(z)$ on the circle $|z| = r$. We know by (17) that $l(z)$ is nonpositive for those z corresponding to the vertical lines of the parallelogram. Hence, its simple zero must be a corner point: either $[\log r + i\gamma]$ or $[\log r + i(\gamma + \pi)]$. Observe that the only other point on the boundary where $L(t)$ can change its sign is the point $\log b$ where it has a simple pole. Thus we infer that on the boundary of the half period parallelogram $L(t)$ is positive on the segment $\langle \log r + i\gamma, \log |b| + i\gamma \rangle$ and is negative on the rest.

Returning to the z -plane, we find that $z^2 k'(z)^2$ is negative for $z = se^{i\gamma}$ with $|b| < s < 1$. Now,

$$\frac{d}{ds} k(e^{i\gamma}s) = e^{i\gamma} k'(e^{i\gamma}s)$$

and hence we can conclude that $d/ds k(e^{i\gamma}s)$ is imaginary for this interval.

Thus $k(b) - k(e^{i\gamma}) = \text{imaginary}$. On the other hand, observe that $J(B)$ is pure imaginary and so is, by (15), $k(b)$. Hence, we conclude $k(e^{i\gamma}) = \text{imaginary}$. Finally, $J(w)$ is real for $|w| = 1$ by (16), whence $k(e^{i\gamma}) = 0$. By (15') the image w of $e^{i\gamma}$ satisfies $J(w) = 0$. By (16), $J(1) = 0$. By the normalization, the point 1 corresponds to 1. Hence, $e^{i\gamma} = 1$, and we infer $b > 0$.

It has been shown that $L(t)$ vanishes at $\log r + i\gamma$ and that $\gamma = 0$. This implies that $k'(r) = 0$.

6. Now we complete the argument as follows: $k(z)$ is pure imaginary on the real axis between b and 1. It follows from (15') and (16) that $w = f(z)$ is real on that segment. On the segment between r and b we see that $l(z) > 0$, which implies $(k'(z)z)^2 > 0$. Hence $k'(z)z$ and $k'(z)$ are real on that segment. Thus (15'') leads to the differential equation

$$\frac{dw}{\sqrt{w(B-w)(1-wB)}} = \text{real}.$$

This implies that $w = f(z)$ is real between r and b .

We conclude that $f(z)$ maps the inner circumference $|z| = r$ onto the continuum Γ which is a segment of the positive axis. The point $z = -r$ is mapped into $w = 0$ and $z = r$ into the right end of the slit.

Thus the single valued analytic function $[zk'(z)]^2$ is negative on the circumferences $|z| = 1$ and $|z| = r$, has a simple pole at $z = b$, and vanishes at $z = r$.

By Rouché's theorem it is easy to see that this function takes every nonnegative value precisely once. Hence it is a univalent function in the circular ring which maps this domain onto the complex plane slit along two segments of the negative axis. In particular, the segment corresponding to the circumference $|z| = r$ begins at the origin.

The geometric properties just described identify $[zk'(z)]^2$ up to a factor as

$$[zk'(z)]^2 = \frac{A}{\wp(\theta) - \wp(\phi)}, \quad (18)$$

where $\theta = \log z/r$ and $\phi = \log b/r$ and $\wp(\theta)$ is the Weierstrass $\wp$ -function of period $\log 1/r$ and $i\pi$ (see, for example, [7, p. 191]).

7. Solving for J yields

$$J[f(z)] = \frac{K+1}{2} [k + \bar{k}Q] \quad (19)$$

where J is the elliptic integral defined in (16) and $Q = \sqrt{(K-1)/(K+1)}$.

After inverting (19) we will have an explicit expression for the extremal function

$$f(z) = P \left\{ \frac{K+1}{2} (k(z) + \overline{k(z)} Q) \right\}, \quad (20)$$

where P is closely related to the Weierstrass $\wp$ -function.

8. Although the analytic function $k(z)$ is determined only up to a factor, various relationships can be used in order to determine the extremal value B . The extremal function is explicitly expressed in (20) as the inverse of the elliptic integral (16). The periods of (20) are functions of B . The branch points of (16) are real.

From (1) it follows that as z traverses the unit circle, w does the same. By Cauchy's integral theorem

$$\int_{|w|=1} \frac{dw}{\sqrt{w(B-w)(1-Bw)}} = 2 \int_0^B \frac{dw}{\sqrt{w(B-w)(1-Bw)}} = 2\omega_1. \quad (21)$$

The right-hand side of (21) is twice the real period of the elliptic integral (16). Consider equation (15) for $|z| = 1$. Since dk is real on that circumference, we find that

$$\int_{|z|=1} k'(z)dz = 2(1-Q)\omega_1. \quad (22)$$

The imaginary period is

$$\omega_2 = \int_B^{1/B} \frac{dw}{\sqrt{w(B-w)(1-Bw)}}. \quad (23)$$

Since

$$\int_B^1 \frac{dw}{\sqrt{w(B-w)(1-Bw)}} = \int_1^{1/B} \frac{dw}{\sqrt{w(B-w)(1-Bw)}},$$

we have $\frac{1}{2}\omega_2 = \int_B^1 dw/\sqrt{w(B-w)(1-Bw)}$. From (15) we have

$$\int_b^1 dk = \int_B^1 dJ - Q \int_B^1 \overline{dJ} = \frac{1}{2}\omega_2 + \frac{Q}{2}\omega_2, \quad (24)$$

from which follows

$$\int_b^1 k'(z)dz = \frac{1}{2}(1+Q)\omega_2. \quad (25)$$

The left sides of (22) and (25) are known up to a factor. The quotient of (25) by (22) is thus a completely determined function of b . Denote the quotient by $E(b)$. The ratio of the periods is then

$$4E(b) \frac{(1-Q)}{(1+Q)} = \frac{\omega_2}{\omega_1} = \tau. \quad (26)$$

9. Since the ratio τ of the periods is known from (26), it is possible to express the extremal value B as a function of the fixed point b . This can be done as follows.

Putting the elliptic integral (16) in the Weierstrass normal form, we have

$$\int \frac{dw}{\sqrt{w(B-w)(1-wB)}} = \int \frac{dw}{\sqrt{4(w-e_1)(w-e_2)(w-e_3)}}$$

where $e_1 + e_2 + e_3 = 0$.

The result of a straightforward calculation is

$$B^2 = \frac{e_2 - e_3}{e_1 - e_3}. \quad (27)$$

But the right side is the elliptic modular function $\lambda(\tau)$ which is a single valued function of τ , [15], [16], which in turn is a function of b .

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