A FIXED POINT THEOREM AND ATTRACTORS

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ABSTRACT. We investigate attractors for compact sets by considering a certain quotient space. The following theorem is included. Let $f: G \rightarrow G$, G a closed convex subset of a Banach space, f a mapping satisfying (i) there exists $M \subset G$ which is an attractor for compact sets under f ; (ii) the family $\{f^n\}_{n=1}^\infty$ is equicontinuous. Then f has a fixed point.

1. Since the appearance of the celebrated Schauder fixed point theorem, the following has been a longstanding open question.

CONJECTURE 1. Let $f: G \rightarrow G$, G a closed, convex subset of a Banach space, f a continuous mapping with f^N compact, $N > 1$. Then f has a fixed point (?)

In a paper which appeared in 1972, Roger Nussbaum [6] states Conjecture 1 along with several new conjectures concerning fixed points. In that paper he introduces the concept of an attractor for compact sets.

DEFINITION 1.1. Let X be a topological space, $f: X \rightarrow X$ a map, and M a nonempty subset of X . M is an attractor for compact sets under f if (1) M is compact and $f(M) \subset M$, and (2) given any compact set $C \subset X$ and any open neighborhood U of M , there exists an integer $N = N(C, U)$ such that $f^n(C) \subset U$ for $n \geq N$. M is an attractor for points under f if (2) holds for $C = \{x\} \subset X$. For brevity, we shall use "a.c.s. under f " for the phrase "attractor for compact sets under f ."

In this setting of Conjecture 1, we see that $\text{cl}(f^N(G))$ is an a.c.s. under f ; hence, an affirmative answer to the following conjecture of Nussbaum would yield an affirmative answer to Conjecture 1.

CONJECTURE 2. Let G be a closed, convex subset of a Banach space X and $f: G \rightarrow G$ a continuous map. If there exists a set $M \subset G$ which is an attractor for compact sets under f , then f has a fixed point (?)

Nussbaum [6] states that he does not think that Conjecture 2 has an affirmative answer. If one attempts to provide a counterexample to Conjecture 2, a characterization of functions which have attractors seems desirable. In the direction of obtaining information concerning the structure of a.c.s., see Solomon [7].

In §2, we provide several results concerning those functions which have a.c.s. In §3, we provide a partial affirmative answer to Conjecture 2 and thus

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Conjecture 1, for functions f satisfying the additional condition that the sequence of iterates, $\{f^n\}$, of f be equicontinuous.

2. A quotient space. Throughout the remainder of this paper, X is assumed to be a metrizable topological space.

DEFINITION 2.1. Let f be a continuous self-map on the metrizable topological space X with M an f -invariant compact subset of X . We denote by X/M the quotient space equipped with the quotient topology arising from X by identifying M to a point. $f^*: X/M \rightarrow X/M$ will denote the induced map which makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \Pi \downarrow & & \downarrow \Pi \\ X/M & \xrightarrow{f^*} & X/M \end{array}$$

commute where Π is the natural projection.

The following facts concerning X/M are easily established.

LEMMA 2.1. (1) If U is an open subset of X and $M \subset U$, then $\Pi(U)$ is open in X/M . (2) Π is a closed map and since $\{\Pi^{-1}(x^*)\}$ is compact for every x^* in X/M , Π is a perfect map.

Employing a well-known result of Hanai [2], the image under a continuous, closed perfect map of a metrizable topological space is metrizable—we obtain the following lemma.

LEMMA 2.2. X/M is metrizable.

LEMMA 2.3. If Y^* is a compact subset of X/M such that $a^* = \Pi(M) \in Y^*$, then $\Pi^{-1}(Y^*)$ is compact.

PROOF. The continuity of Π asserts that $\Pi^{-1}(Y^*)$ is closed in X . Let $\{U_\alpha\}$ be an open covering of $\Pi^{-1}(Y^*)$. Since $M \subset \Pi^{-1}(Y^*)$ and M is compact, there exist indices $\alpha_1, \alpha_2, \dots, \alpha_m$ such that $M \subset \bigcup_{i=1}^m U_{\alpha_i}$. Let $W = \bigcup_{i=1}^m U_{\alpha_i}$ and define the open covering $\{V_\alpha\}$ by $V_\alpha = U_\alpha \cup W$. Note that $M \subset V_\alpha$ for each α which implies that $\Pi(V_\alpha)$ is open in X/M . Since $\{\Pi(V_\alpha)\}$ covers Y^* , there exists a finite subcovering $\Pi(V_{\beta_1}), \Pi(V_{\beta_2}), \dots, \Pi(V_{\beta_n})$ which covers Y^* . Since $\Pi^{-1}(\Pi(V_\alpha)) = V_\alpha$ for each α , this implies that $\{V_{\beta_i}\}_{i=1}^n$ covers $\Pi^{-1}(Y^*)$ which in turn implies that the collection $\{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_m}, U_{\beta_1}, U_{\beta_2}, \dots, U_{\beta_n}\}$ covers $\Pi^{-1}(Y^*)$; thus, $\Pi^{-1}(Y^*)$ is compact.

LEMMA 2.4. If M is an attractor for compact sets under f and $x^* \in X/M$, then the sequence $\{f^{*n}(x^*)\}$ converges to a^* .

PROOF. If $x^* = a^*$ the statement is clearly true since $f^{*n}(a^*) = a^*$ for each n . If $x^* \neq a^*$ then $\Pi^{-1}(x^*)$ is a singleton, $\{x\}$, hence a compact set in X . Choosing an arbitrary open set U^* in X/M containing a^* , we obtain an open set $\Pi^{-1}(U^*)$ in X containing M . Since M is an a.c.s. under f , there is a

positive integer N such that $f^n(x) \in \Pi^{-1}(U^*)$ for each $n \geq N$; hence $f^{*n}(x^*) \in U^*$ for each $n \geq N$.

LEMMA 2.5. *If M is an a.c.s. under f and Y^* is a nonempty, f^* -invariant compact subset of X/M , then $\bigcap_{n=1}^{\infty} f^{*n}(Y^*) = \{a^*\}$.*

PROOF. Since $f(Y^*) \subset Y^*$, Lemma 2.4 implies that $a^* \in Y^*$ and Lemma 2.3 implies that $\Pi^{-1}(Y^*)$ is compact. Let U^* be any open set in X/M containing a^* . Since M is an a.c.s. under f there is a positive integer N such that $f^n(\Pi^{-1}(Y^*)) \subset \Pi^{-1}(U^*)$ for $n \geq N$, implying $f^{*n}(Y^*) \subset U^*$. Since U^* was arbitrary, our lemma follows.

We now provide a statement which relates attractors in X and X/M .

THEOREM 2.1. *Let f be a continuous self-map on a metrizable topological space X and M a nonempty compact f -invariant subset of X . Then M is an attractor for compact sets under f if and only if $\{a^*\}$, with $a^* = \Pi(M)$, is an attractor for compact sets under the induced mapping f^* .*

PROOF. Let M be an a.c.s. under f and C^* any compact set in X/M , then $C^* \cup \{a^*\}$ is also compact. If O^* is any open neighborhood of a^* , then $\Pi^{-1}(O^*)$ is open in X and contains M . From Lemma 2.3, $\Pi^{-1}(C^* \cup \{a^*\})$ is compact, thus there exists N , a positive integer, such that $f^n(\Pi^{-1}(C^* \cup \{a^*\})) \subset \Pi^{-1}(O^*)$ for each $n \geq N$ implying that $f^{*n}(C^* \cup \{a^*\}) \subset O^*$; hence, $\{a^*\}$ is an a.c.s. under f^* .

Conversely, let us assume that $\{a^*\}$ is an a.c.s. under f^* . Let C be any compact set in X . If O is any open set containing M , we consider the compact set $\Pi(C)$ and the open set $\Pi(O)$ in X/M containing $\{a^*\}$. Since $\{a^*\}$ is an attractor for compact sets under f^* , there is a positive integer N such that $f^{*n}(\Pi(C)) \subset \Pi(O)$ for $n \geq N$; thus, it follows that $f^n(C) \subset O$ showing that M is an a.c.s. under f .

Let (Y, d) be a metric space and f a self-map on Y . We say that f is contractive in the sense of Edelstein if for each y_1, y_2 in Y , with $y_1 \neq y_2$, we have that $d(f(y_1), f(y_2)) < d(y_1, y_2)$.

In the sequel we shall need the following result of Janos [4].

THEOREM 2.2. *If f is a continuous self-map on a metrizable topological space X such that $\{f^n(x)\}_{n=1}^{\infty}$ converges for each x in X , then the following statements are equivalent:*

- (1) *X can be metrized in such a way that f becomes contractive in the sense of Edelstein;*
- (2) *For a nonempty compact f -invariant subset Y of X , the set $\bigcap_{n=1}^{\infty} f^n(Y)$ is a singleton.*

By employing Lemma 2.4, Lemma 2.5, and Theorem 2.2, we have the following.

THEOREM 2.3. *If M is an a.c.s. under $f: X \rightarrow X$, then there exists a metric d^* ,*

compatible with the topology on X/M , relative to which the induced map f^* is contractive in the sense of Edelstein.

To see that the converse of Theorem 2.3 is not in general true, we provide the following.

EXAMPLE 1. Let B denote the closed unit disc in the real Hilbert space l_2 . Define $f: B \rightarrow B$ as follows. For $x = (x_1, x_2, \dots) \in B$, $f(x) = (0, \alpha_1 x_1, \alpha_2 x_2, \dots)$, where $\alpha_i = 1 - 1/(i+1)^2$, $i = 1, 2, \dots$. Since f is a linear self-map and for any arbitrary but fixed nonzero x in B we have, letting N denote the smallest integer such that $x_N \neq 0$, that

$$\begin{aligned}\|f(x)\|^2 &= \alpha_N^2 |x_N|^2 + \sum_{i=N+1}^{\infty} \alpha_i^2 |x_i|^2 \\ &\leq \alpha_N^2 |x_N|^2 + \|x\|^2 - |x_N|^2 = \|x\|^2 - (1 - \alpha_N^2) |x_N|^2\end{aligned}$$

or

$$\|f(x)\|^2 < \|x\|^2,$$

implying $\|f(x)\| < \|x\|$; i.e., f is contractive. We also observe that

$$\|f^n(x)\|^2 \geq \left(\prod_{i=0}^{n-1} \alpha_{N+i} \right) |x_N|^2 > \frac{N}{N+1} |x_N|^2,$$

since

$$\prod_{i=N}^{N+k} \alpha_i = \prod_{i=1}^{N+k} \alpha_i / \prod_{i=1}^{N-1} \alpha_i = \frac{(N+k+2)N}{(N+k+1)(N+1)} > \frac{N}{N+1}, \quad \text{for } k \geq 0.$$

Hence, $f^n(x) \not\rightarrow 0$, the unique fixed point, as $n \rightarrow \infty$. Thus, the existence of a unique fixed point x_0 with f a contractive mapping is not sufficient to conclude that $\{x_0\}$ is an attractor for compact sets under f .

3. A fixed point theorem. In 1965, Felix Browder [1] and W. A. Kirk [5] produced, independently, a fixed point theorem for nonexpansive self-maps defined on a closed convex subset of a Banach space. Both relied heavily upon the concept of normal structure.

The conditions of the theorem which we shall present are less stringent in one respect, namely, the nonexpansiveness of f is relaxed to the assumption of equicontinuity of the sequence of iterates $\{f^n\}$ and the normal structure is not assumed, but we postulate the existence of an attractor for compact sets under f .

REMARK 3.1. If $f: X \rightarrow X$ is a nonexpansive self-map on a metric space (X, d) , then the family of iterates $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. If X is compact, the equicontinuity of $\{f^n\}$ is implied by assuming that for some $N \geq 1$, the self-map f^N is nonexpansive.

The proof of our theorem makes use of the following fact (see [3]).

LEMMA 3.1. Let (X, d) be a compact metric space and $f: X \rightarrow X$ with $\{f^n: n \geq 1\}$ equicontinuous. Then there exists a retraction $r: X \rightarrow C_f$ where C_f

denotes the core of f ; i.e., $C_f = \bigcap \{f^n(X) : n \geq 1\}$.

THEOREM 3.1. *Let $f: G \rightarrow G$, G a closed convex subset of a Banach space, f a mapping satisfying*

- (i) *there exists $M \subset G$ which is an attractor for compact sets under f , and*
- (ii) *the family $\{f^n\}_{n=1}^\infty$ is equicontinuous (relative to the metric induced by the norm). Then f has a fixed point.*

PROOF. Let Y denote the closed convex hull of M ; i.e., $Y = \text{cl}[\text{co}(M)]$. Thus, Y is a nonempty compact convex subset of G , not necessarily f -invariant. Denote by X the closure of $\bigcup_{n=0}^\infty f^n(Y)$ ($f^0(x) = x$). We shall now show that X is f -invariant and compact. Since $f[f^n(Y)] = f^{n+1}(Y)$ for every $n \geq 0$, it follows that $\bigcup_{n=0}^\infty f^n(Y)$ is f -invariant. The continuity of f yields that X is f -invariant.

To prove that X is compact, we show that $\bigcup_{n=0}^\infty f^n(Y)$ is totally bounded. Given $\varepsilon > 0$, we consider the $\varepsilon/2$ open neighborhood of M which we denote by $M_{\varepsilon/2}$ —if d denotes the metric induced by the norm and $\alpha > 0$, $M_\alpha = \{x \in G : d(x, M) < \alpha\}$. From the definition of an a.c.s. under f , we have that there exists a positive integer N_0 such that $n \geq N_0$ implies that $f^n(Y) \subset M_{\varepsilon/2}$. Since M is compact, it has a finite $\varepsilon/2$ -mesh implying that $M_{\varepsilon/2}$ has a finite ε -mesh. Since $\bigcup_{n=0}^{N_0} f^n(Y)$ is compact; i.e., has a finite ε -mesh, this implies that $\bigcup_{n=0}^\infty f^n(Y)$ has a finite ε -mesh. Since ε was arbitrary and X is complete we have that X is compact.

Let us denote by C the core of the restriction of f to X ; i.e., $C = \bigcap_{n=0}^\infty f^n(X)$. It follows that C is nonempty, compact, and f -invariant. C being a compact f -invariant subset of G and M being an a.c.s. under f implies that $C \subset M$.

Since f is equicontinuous on X , we have by Lemma 3.1, that there exists a retraction $r: X \rightarrow C$. Recalling that $C \subset Y \subset X$, we can consider the self-mapping $f \circ r: Y \rightarrow Y$. Since Y is compact, convex, and $f \circ r$ is continuous, the Schauder fixed point theorem ensures the existence of a fixed point for $f \circ r$. Thus there is a $y \in Y$ such that $f \circ r(y) = y$. But since $r(y) \in C$ and C is f -invariant, we have that $y \in C$, which implies, since r is the identity on C , that $r(y) = y$ which in turn furnishes the desired result $f(y) = y$.

REMARK. Our proof actually does not use the full strength of part (ii) of the hypothesis, but only the weaker assumption: (ii)* M contains an f -invariant retract of X ; i.e., there is some $C^* \subset M$ such that there is a retraction $r: X \rightarrow C^*$ and $f(C^*) \subset C^*$. Since $C^* \subset Y$ we again consider the self-map $f \circ r: Y \rightarrow Y$ and our proof goes through.

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