BRAUER GROUPS OF LINEAR ALGEBRAIC GROUPS WITH CHARACTERS

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ABSTRACT. Let G be a connected linear algebraic group over an algebraically closed field of characteristic zero. Then the Brauer group of G is shown to be $C \times (\mathbf{Q}/\mathbf{Z})^{(n)}$ where C is finite and $n = d(d-1)/2$, with d the $\mathbf{Z}$ -rank of the character group of G . In particular, a linear torus of dimension d has Brauer group $(\mathbf{Q}/\mathbf{Z})^{(n)}$ with n as above.

In [6], B. Iversen calculated the Brauer group of a connected, characterless, linear algebraic group over an algebraically closed field of characteristic zero: the Brauer group is finite—in fact, it is the Schur multiplier of the fundamental group of the algebraic group [6, Theorem 4.1, p. 299]. In this note, we extend these calculations to an arbitrary connected linear group in characteristic zero. The main result is the determination of the Brauer group of a d -dimensional affine algebraic torus, which is shown to be $(\mathbf{Q}/\mathbf{Z})^{(n)}$ where $n = d(d-1)/2$. (This result is noted in [6, 4.8, p. 301] when $d = 2$.) We then show that if G is a connected linear algebraic group whose character group has $\mathbf{Z}$ -rank d , then the Brauer group of G is $C \times (\mathbf{Q}/\mathbf{Z})^{(n)}$ where n is as above and C is finite.

We adopt the following notational conventions: F is an algebraically closed field of characteristic zero, and $T = (F^*)^{(d)}$ is a d -dimensional affine torus over F . We use $H_{\text{ét}}^*$, H_{cl}^* , and H_{gr}^* to denote étale, singular, and group cohomology. If G is an abelian group and m a positive integer, ${}_mG$ denotes the m -torsion in G . If X is an affine F -variety, $F[X]$ is its coordinate ring. $\text{Br}(\cdot)$ denotes Brauer group and $[\cdot]$ denotes the class in the Brauer group of an Azumaya algebra. $X(G)$ is the character group of the algebraic group G and $U(A)$ denotes the units group of the ring A . We let G_m denote $GL_1(F)$.

PROPOSITION 1. *Let A be a finite abelian group and let X be a smooth F -variety such that $H_{\text{ét}}^i(X, A) = 0$ for $i = 1, 2$. Then $H_{\text{ét}}^2(X \times T, A) = A^{(n)}$ where $n = d(d-1)/2$.*

PROOF. By the Lefschetz principle and smooth base change [1, Corollary 1.6, p. 211] we may assume $F = \mathbf{C}$. By the comparison theorem for classical and étale cohomology [1, Theorem 4.4, p. 74]

$$H_{\text{ét}}^2(X \times T, A) = H_{\text{cl}}^2(X \times T, A).$$

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Let $V = \mathbb{C}^{(d)}$ and let $\Gamma = Z^{(d)}$. Then $T = V/\Gamma$. Let $Y = X \times V$. Then Γ operates on Y with $Y/\Gamma = X \times T$. Since Y is homotopically equivalent to X , $H_{\text{cl}}^i(Y, A) = H_{\text{cl}}^i(X, A)$, so by the comparison theorem $H_{\text{cl}}^i(Y, A) = 0$ for $i = 1, 2$. The cohomology spectral sequence of a covering then yields a spectral sequence

$$H_{\text{cl}}^p(\Gamma, H_{\text{cl}}^q(Y, A)) \Rightarrow H_{\text{cl}}^n(X \times T, A),$$

with $E_{\infty}^{2,0} = H_{\text{gr}}^2(\Gamma, A)$ and $E_{\infty}^{1,1} = E_{\infty}^{0,2} = 0$, so $H_{\text{cl}}^2(X \times T, A) = H_{\text{gr}}^2(\Gamma, A)$. By [7, p. 189], $H_{\text{gr}}^2(\Gamma, A)$ is of the desired form.

COROLLARY 2. *Let $\bar{G}$ be a simply connected linear algebraic group over F . Then*

$${}_m H_{\text{et}}^2(\bar{G} \times T, G_m) = (Z/mZ)^{(n)}$$

where $n = d(d-1)/2$.

PROOF. By [4, Corollary 4.4, p. 278], $\text{Pic}(\bar{G}) = 1$, and hence $\text{Pic}(\bar{G} \times T) = 1$. Then the Kummer sequence for m ,

$$0 \rightarrow Z/mZ \rightarrow G_m \rightarrow G_m \rightarrow 1$$

shows that ${}_m H_{\text{et}}^2(\bar{G} \times T, G_m) = H_{\text{et}}^2(\bar{G} \times T, Z/mZ)$ and ${}_m H_{\text{et}}^2(\bar{G}, G_m) = H_{\text{et}}^2(\bar{G}, Z/mZ)$. By [6, Theorem 4.1, p. 299], ${}_m H_{\text{et}}^2(\bar{G}, G_m) = 1$. Also,

$$H_{\text{et}}^1(\bar{G}, Z/mZ) = \text{Hom}(\pi_1 \bar{G}, Z/mZ) = 1$$

since $\bar{G}$ is simply connected. Proposition 1 now implies the result.

COROLLARY 3. *Let $\bar{G}$ be a simply connected linear algebraic group over F . Then*

$$H_{\text{et}}^2(\bar{G} \times T, G_m) = (\mathbb{Q}/Z)^{(n)}$$

where $n = d(d-1)/2$. In particular, $H_{\text{et}}^2(T, G_m) = (\mathbb{Q}/Z)^{(n)}$.

PROOF. The first assertion follows from Corollary 2 and the fact that $H_{\text{et}}^2(\bar{G} \times T, G_m)$ is torsion [5, Proposition 1.4, p. 71]. The second is the special case where $\bar{G} = \{e\}$.

In the notation of Corollary 3, there is a canonical injection

$$\text{Br}(\bar{G} \times T) \rightarrow H_{\text{et}}^2(\bar{G} \times T, G_m)$$

[5, Proposition 1.4, p. 48]. We next construct Azumaya algebras to show that the injection is onto.

DEFINITION. Let R be a domain with $1/m \in R$ and assume R contains a primitive m th root of unity w . Let a and b be units of R . Then $A_R^w(a, b)$ denotes the associative R -algebra with identity generated by two elements x and y subject to the relations: $x^m = a$, $y^m = b$, $yx = wxy$. We note that $A_R^w(a, b)$ is a free R -module of rank m^2 .

LEMMA 4. *Let R be a domain. Assume $1/m \in R$ and that R contains a*

primitive m th root of unity w . Let a and b be units of R . Then $A_R^w(a, b)$ is an Azumaya R -algebra.

PROOF. Let $A = A_R^w(a, b)$, let M be a maximal ideal of R , and let $S = R/M$. Then $A/MA = A_S^w(a, b)$, where w, a, b represent the corresponding images in S . Since $1/m \in S$, w is still a primitive m th root of unity in S , and hence by [8, Theorem 15.1, p. 144], A/MA is a central simple S -algebra. Since A is free as R -module and A/MA is central simple over R/M for every maximal ideal M , A is an Azumaya R -algebra by [2, Theorem 4.7, p. 379].

LEMMA 5. Let $R = F[t_1, t_1^{-1}, t_2, t_2^{-1}]$, and let w be a primitive m th root of unity in F . Then $A_R^w(t_1, t_2)$ has order m in $\text{Br}(R)$.

PROOF. Let $A = A_R^w(t_1, t_2)$ and let $K = F(t_1, t_2)$. Then $\text{Br}(R) \rightarrow \text{Br}(K)$ is an injection by [2, Theorem 7.2, p. 388]. Thus it suffices to compute the order of $a = [A \otimes_R K] = [A_K^w(t_1, t_2)]$ in $\text{Br}(K)$. By [8, Theorem 15.7, p. 149], $[A_K^w(c, b)] = 1$ if and only if c is a norm from $K(\sqrt[m]{b})$ to K , and by [8, Theorem 15.1, p. 144], $a^k = [A_K^w(t_1^k, t_2)]$. It follows that $a^m = 1$. Suppose $a^k = 1$ for $1 \leq k < m$. $K(\sqrt[m]{t_2}) = F(t_1, x)$ where $x^m = t_2$. If t_1^k is a norm from $F(t_1, x)$, there are relatively prime polynomials $p, q \in F[t_1, x]$ with

$$t_1^k = \prod_{i=0}^{m-1} p(t_1, w^i x) q(t_1, w^i x)^{-1},$$

so $t_1^k \prod q(t_1, w^i x) = \prod p(t_1, w^i x)$. Then $t_1 | p(t_1, w^i x)$ for some i , and hence for all i , so t_1^m divides $t_1^k \prod q(t_1, w^i x)$. It follows that t_1^{m-k} divides $q(t_1, w^i x)$ for some i , and hence for all i . In particular, $t_1 | p$ and $t_1 | q$, contrary to choice of p and q . Thus there is no such k and the result follows.

THEOREM 6. Let w be a primitive m th root of unity in F , let $t_1, \dots, t_d$ be a Z -basis of $X(T)$, and let $R = F[T]$. Then $\{[A_R^w(t_i, t_j)] \mid 1 \leq i < j \leq d\}$ is a Z/mZ -basis of ${}_m H_{\text{et}}^2(T, G_m)$, and ${}_m \text{Br}(T) = {}_m H_{\text{et}}^2(T, G_m)$.

PROOF. Let $a(i, j)$ denote the class $[A_R^w(t_i, t_j)]$ in $\text{Br}(R)$. Suppose $x = \prod a(i, j)^{n(i, j)} = 1$ in $\text{Br}(R)$. Now $R = F[t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}]$. Fix i, j . Let $f: R \rightarrow F[t_i, t_i^{-1}, t_j, t_j^{-1}]$ be defined by $f(t_k) = 1$ if $k \neq i$ or $k \neq j$. Then

$$\text{Br}(f)(x) = a(i, j)^{n(i, j)}$$

(in the obvious notation). By Lemma 5, $m | n(i, j)$. Thus the $a(i, j)$ generate a subgroup of $\text{Br}(R)$ of order $md(d-1)/2$. By Corollary 2 with $\bar{G} = \{e\}$, ${}_m H_{\text{et}}^2(T, G_m)$ has the same order, and the result follows.

COROLLARY 7. $\text{Br}(T) = H_{\text{et}}^2(T, G_m) = (\mathbf{Q}/Z)^{(n)}$ where $n = d(d-1)/2$.

PROOF. Apply Theorem 6 and Corollary 3.

The isomorphism of Theorem 6 has the following invariant description: fix an m th root of unity w in F . Then the bilinearity [8, p. 146] and skew-symmetry [8, p. 147] and [8, p. 94] of the A_R^w -construction shows that there is a group homomorphism

$$\Lambda^2(X(T)) \otimes_Z Z/mZ \rightarrow {}_m\text{Br}(T)$$

given by $(u \wedge v) \otimes k = A^w(u, v)^k$, and by Theorem 6 this is an isomorphism.

This isomorphism is natural in T . Consider an isogeny $\phi: T \rightarrow T$ given by

$$\phi(x_1, \dots, x_d) = (x_1^{e_1}, \dots, x_d^{e_d}).$$

Let $t_1, \dots, t_d$ be the Z -basis of $X(T)$ given by $t_i(x_1, \dots, x_d) = x_i$. Then the induced map

$$\phi^*: \Lambda^2(X(T)) \rightarrow \Lambda^2(X(T))$$

sends $t_i \wedge t_j$ to $e_i e_j (t_i \wedge t_j)$, and there is an exact sequence

$$0 \rightarrow \Lambda^2(X(T)) \rightarrow \Lambda^2(X(T)) \rightarrow C \rightarrow 0$$

where $C = \prod \{Z/e_i e_j Z \mid 1 \leq i < j \leq d\}$. By standard techniques we can identify the kernel of $\phi^* \otimes Z/mZ$ with $\text{Tor}_1(C, Z/mZ) = {}_m C$. Thus the kernel of $\text{Br}(\phi)$ on ${}_m\text{Br}(T)$ is isomorphic to ${}_m C$. We further observe that if $e_1 = \dots = e_d = m$, then $\text{Br}(\phi)$ is the zero map on ${}_m\text{Br}(T)$, so every m -torsion element of $\text{Br}(T)$ becomes trivial under the isogeny $T \rightarrow T$ which sends x to x^m .

The following lemma seems to be known, but for lack of a suitable reference we include a proof.

LEMMA 8. *Let P be a reductive connected linear algebraic group over F . Then there is a torus T in P of dimension equal to the rank of $X(P)$ such that $P = (P, P) \times T$ as varieties.*

PROOF. Let T_1 be a maximal torus of $S = (P, P)$ and let T_2 be a maximal torus of P containing T_1 . There is a subtorus T of T_2 such that $T_2 = T_1 \times T$. T commutes with T_1 and T_1 is its own centralizer in S , so $S \cap T = \{e\}$. Also, $P = ST_2$, so $P = ST = S \times T$ (as varieties).

THEOREM 9. *Let G be a connected linear algebraic group over F . Let P be a maximal reductive subgroup of G and let Π be the fundamental group of (P, P) . Then $\text{Br}(G) = W \times \Pi^{(d)} \times (Q/Z)^{(n)}$, where W is the Schur multiplier of Π , d is the rank of $X(G)$ and $n = d(d-1)/2$.*

PROOF. As varieties, $G = U \times P$ where U is the unipotent radical of G . Since $F[U]$ is a polynomial ring, by [2, Proposition 7.7, p. 391], $\text{Br}(G) = \text{Br}(P)$, so we may assume $G = P$ is reductive. We note that $X(G)$ and $X(P)$ have the same rank. Let $S = (P, P)$ and write $P = S \times T$ as in Lemma 8. Let $\bar{S}$ be the simply connected covering group of S , and let $\bar{P} = \bar{S} \times T$. Since $\bar{S}/\Pi = S$, $\bar{P}$ is an étale covering space of P with group $\Pi \times 1 = \Pi$. By Corollary 3 and Theorem 6, $T \rightarrow \bar{P}$ induces an isomorphism $\text{Br}(T) \xrightarrow{\sim} \text{Br}(\bar{P})$. Let $p: \bar{P} \rightarrow P$ be the covering map and let $a \in \text{Br}(P)$. Then by the above there is an $x \in \text{Br}(T)$ such that $\text{Br}(p)(ax) = 1$, so $\text{Br}(P) = \text{Br}(T)\text{Br}(\bar{P}/P)$. To compute $\text{Br}(\bar{P}/P)$, we note that $\text{Pic}(\bar{P}) = \text{Pic}(\bar{S}) = 1$ [4, Corollary 4.4, p. 278]. It follows from [3, Corollary 5.5, p. 17], that $\text{Br}(\bar{P}/P) = H_{\text{gr}}^2(\Pi, G_m(\bar{P}))$. Now

$$G_m(\bar{P}) = U(F[\bar{P}]) = U(F[\bar{S}][t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}])$$

and by [4, Corollary 2.2, p. 273], $U(F[\bar{S}]) = F^*$. It follows that $G_m(T) \rightarrow G_m(\bar{P})$ is an isomorphism. Let $V = G_m(T)$. We then have a split exact sequence of Π -modules

$$1 \rightarrow F^* \rightarrow V \rightarrow Z^{(d)} \rightarrow 1.$$

Thus $\text{Br}(\bar{P}/P) = H_{\text{gr}}^2(\Pi, V) = H_{\text{gr}}^2(\Pi, F^*) \times H_{\text{gr}}^2(\Pi, Z^{(d)})$. Now $H_{\text{gr}}^2(\Pi, F^*)$ is the Schur multiplier of Π and $H_{\text{gr}}^2(\Pi, Z)$ is the character group of Π , and the latter is isomorphic to Π since Π is abelian. Thus $\text{Br}(\bar{P}/P) = W \times \Pi^{(d)}$. Since $\text{Br}(T) \cap \text{Br}(\bar{P}/P) = 1$, $\text{Br}(P) = W \times \Pi^{(d)} \times \text{Br}(T)$, and the theorem follows from Corollary 7.

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