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MOIŠEZON SPACES AND POSITIVE COHERENT SHEAVES[1]

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ABSTRACT. In recent papers of Grauert and Riemenschneider, attempts have been made to generalize Kodaira's embedding theorem to a characterization of Moišezon spaces. In this paper, we define a torsion-free coherent analytic sheaf of generic fiber dimension one as positive if its monoidal transform is positive. We prove: a normal irreducible compact complex space is Moišezon if and only if it carries a positive coherent sheaf of generic fiber dimension one.

0. The object of this paper is to characterize Moišezon spaces by positive coherent sheaves. A Moišezon space is a normal irreducible compact complex space whose field of meromorphic functions has trancendence degree equal to the complex dimension of the space. Moišezon proved (see **[7]**, **[8]**, **[9]**) that such spaces have projective desingularizations. Grauert, in **[3]**, associates to a coherent analytic sheaf a dual geometric object which he calls a 'linear space'. This extends to coherent sheaves the duality between the (locally-free) sheaf of sections of a vector bundle and the dual vector bundle. Linear spaces are studied further by Fischer in **[1]** and **[2]**. Grauert and Riemenschneider, in **[4]**, define a coherent sheaf as quasi-positive if the associated linear space carries a fiber metric which restricted to generic fibers (those where the sheaf is locally free) has negative definite curvature almost everywhere. They conjecture that a normal irreducible compact complex space is Moišezon if and only if it carries a quasi-positive coherent sheaf. Partial results appear in **[4]** and in later work of Riemenschneider (**[11]**, **[12]**): every Moišezon space carries a quasi-positive sheaf; an irreducible compact Kähler manifold carrying a quasi-positive sheaf is Moišezon (and hence projective); a normal irreducible compact complex space carrying a quasi-positive coherent analytic sheaf which is actually positive except at isolated points is Moišezon. Related results are discussed by Morrow and Rossi in **[10]** and the conjecture is studied by Wells in **[14]**. The author is indebted to Professors M. Kuranishi, B. G. Moišezon, and R. Ephraim for several useful conversations during his work on this paper.

In this work, we modify the definition of positivity/negativity as follows: Let $E \to X$ be a linear space. Then (cf. **[13**, p. 68**]**) there is a proper analytic

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set $D \subset X$ such that $E|X - D$ is a vector bundle. The primary component of E, denoted $\mathrm{pr}(E)$, is defined as the closure in E of $E|X - D$. ($\mathrm{pr}(E)$ is a linear space in the sense of [3] but not necessarily in that of [1].) Now $\mathbf{C}^*$ acts on $\mathrm{pr}(E) - \{\text{zero-section}\}$ and the equivalence classes form a projective fiber space $P(E)$ associated to the linear space E (see [2, p. 55]). Furthermore, E induces, canonically, a line bundle $L(E)$ over $P(E)$ and any fiber metric on E induces a fiber metric on $L(E)$. A linear space will be called negative if the induced metric on $L(E)$ has negative definite curvature; i.e. E is negative if it carries a metric such that the induced metric on $L(E)$ is given, locally, by a positive function ρ on $P(E)$ with $\ln \rho$ strictly plurisubharmonic. A coherent sheaf is called positive if the associated linear space is negative. Then:

THEOREM. *A normal irreducible compact complex space is Moišezon if and only if it carries a positive coherent sheaf of generic fiber dimension one.*

1. Let X be a complex space and $\Sigma \to X$ a coherent sheaf. Then there exists the monoidal transformation of X with respect to Σ. That is, there is a complex space X_Σ and a proper modification ϕ_Σ: $X_\Sigma \to X$ such that (1) $\Sigma \circ \phi_\Sigma =_{\mathrm{def}} \phi_\Sigma^* \Sigma/\mathrm{Tor}(\phi_\Sigma^* \Sigma)$ is locally free and (2) if ϕ: $Y \to X$ is any proper modification such that $\Sigma \circ \phi$ is locally free then there exists a unique holomorphic map ψ: $Y \to X_\Sigma$ such that $\phi = \phi_\Sigma \circ \psi$ (see [11, p. 268]).

PROPOSITION. *Let X be a complex space and $\Sigma \to X$ a coherent analytic sheaf of gfd (generic fiber dimension) one. Then $X_\Sigma = P(V(\Sigma))$.*

PROOF. Let U be an open set in X over which there exists a resolution $\theta_X^q \to \Sigma \to 0$. By duality, we get $0 \to V(\Sigma) \to U \times \mathbf{C}^q$. Let D be the subvariety of X over which Σ is not locally free (see [13, p. 68]) and let $U_R = U - (U \cap D)$. $V(\Sigma)|U_R$ is a line subbundle of the trivial bundle $U_R \times \mathbf{C}^q$. Let λ: $U_R \to U \times \mathbf{P}^{q-1} = U \times P(\mathbf{C}^q)$ be the map $x \mapsto (x, V(\Sigma)_x)$ where $V(\Sigma)_x$ is the line in $\mathbf{C}^q$ determined by the fiber of $V(\Sigma)$ at x. $X_\Sigma|U$ is, by definition, the closure in $U \times \mathbf{P}^{q-1}$ of the image of λ. Thus we have $0 \to X_\Sigma|U \to U \times \mathbf{P}^{q-1}$. Clearly $X_\Sigma \subseteq P(V(\Sigma))$, for the latter is closed in $U \times \mathbf{P}^{q-1}$. To prove equality, it suffices to show that the closure in $U \times \mathbf{C}^q$ of $V(\Sigma)|U_R$ is $\mathrm{pr}(V(\Sigma))$. But this is precisely the definition of the primary component; the proof is completed.

2. We now state our main theorem.

THEOREM. *A normal irreducible compact complex space is Moišezon if and only if it carries a positive coherent sheaf of gfd one.*

PROOF. Suppose first that X carries a positive sheaf Σ. Then, by definition, the line bundle $L(V(\Sigma)) \to P(V(\Sigma))$ is negative. By Kodaira's theorem, as generalized by Grauert [3, p. 343], $P(V(\Sigma))$ is projective (see also [10, p. 171]). By the above proposition, $P(V(\Sigma))$ is a proper modification of and meromorphically equivalent to X. Thus X is meromorphically equivalent to a projective variety and is consequently Moišezon. Now let X be an arbitrary

Moišezon space. Then there exists a proper modification $\pi: \hat{X} \to X$ where $\hat{X}$ is a projective manifold and the fibers of π are connected. Let E be a very ample line bundle over $\hat{X}$. The global sections of E generate all its fibers and separate points. We have an exact sequence $0 \to E^* \to (\Gamma(E))^*$ which holomorphically embeds $\hat{X}$ into the projective space of lines in the complex vector space $(\Gamma(E))^*$ via the map sending $x \in \hat{X}$ to E_x^*. For future reference, we call this map μ. We fix an identification of $(\Gamma(E))^*$ with $\mathbf{C}^q$ where $q = \dim_{\mathbf{C}}(\Gamma(E)^*)$. Then $0 \to E^* \to \hat{X} \times \mathbf{C}^q$ is exact and the trivial metric in $\hat{X} \times \mathbf{C}^q$ induces a metric in E^* with negative definite curvature (see [**5**, p. 201]). Let π_* and π^* denote direct and inverse image of sheaves, respectively. The exact sequence $\theta_{\hat{X}}^q \to \mathbf{E} \to 0$ (where $\mathbf{E}$ is the sheaf of sections of E) induces a map $\pi_*(\theta_{\hat{X}}^q) \to \pi_*(\mathbf{E})$. Let Σ be the image sheaf of this map. Since X is normal and π has connected fibers, $\pi_*(\theta_{\hat{X}}^q) = \theta_X^q$ so there is a surjection $\theta_X^q \to \Sigma \to 0$. By duality, we have $0 \to V(\Sigma) \to X \times \mathbf{C}^q$. We shall see that the trivial metric in $\mathbf{C}^q$ induces a metric of negative curvature on $L(V(\Sigma))$ and that Σ is therefore a positive sheaf. (This is the same sheaf considered by Riemenschneider; see [**11**, Theorem 4].) We note that Σ is a subsheaf of the torsion-free sheaf $\pi_*(\mathbf{E})$ and is therefore torsion-free.

We claim first that $\Sigma \circ \pi = \mathbf{E}$. For the exact sequence $\theta_X^q \to \Sigma \to 0$ induces an exact sequence $\pi^*\theta_X^q = \theta_{\hat{X}}^q \to \pi^*\Sigma \to 0$. Thus we have exact sequences:

$$0 \to K_\alpha \to \theta_{\hat{X}}^q \xrightarrow{\alpha} \pi^*\Sigma \to 0$$

and

$$0 \to K_\beta \to \theta_{\hat{X}}^q \xrightarrow{\beta} \mathbf{E} \to 0$$

where K_α and K_β are the kernels of α and β, respectively. Let $\hat{D} \subseteq \hat{X}$ be the set on which π is degenerate. Then the two sequences agree on the complement of $\hat{D}$ and thus β vanishes on K_α on the complement of $\hat{D}$. If σ is any local section of K_α, then $\beta(\sigma) \subseteq \mathbf{E}$ is supported on $\hat{D}$. But $\mathbf{E}$ is locally free so that $\beta(\sigma)$ must be zero. Thus $K_\alpha \subseteq K_\beta$ and the identity map on $\theta_{\hat{X}}^q$ induces a surjection $\pi^*\Sigma \to \mathbf{E} \to 0$. The kernel of this map is supported on $\hat{D}$ so it must be precisely $\operatorname{Tor}(\pi^*\Sigma)$. Thus we have:

$$0 \to \operatorname{Tor}(\pi^*\Sigma) \to \pi^*\Sigma \to \mathbf{E} \to 0$$

which means precisely that $\Sigma \circ \pi = \mathbf{E}$ (cf. [**11**, Proposition 3]).

We next claim that in fact $\hat{X}$ is the monoidal transformation of X with respect to Σ. For let X' be this transform. Since $\Sigma \circ \pi$ is locally free and π is a proper modification there is a holomorphic map ψ making the following diagram commute:

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\psi} & X' \\ & \searrow^{\pi} & \downarrow{\scriptstyle \phi'} \\ & & X \end{array}$$

Now π and ϕ' are proper maps so it follows by elementary topology that ψ is a proper map. X' is irreducible (since X is) and, by the proper mapping theorem of Remmert, $\psi(\hat{X})$ is an analytic subvariety of X'. Clearly, $\psi(\hat{X})$ does not have lower dimension than X'. Thus, $\psi(\hat{X}) = X'$, so ψ is surjective. The injectivity of ψ follows from its construction. In fact, X' is contained in $X \times \mathbf{P}^{q-1}$ and $\psi: \hat{X} \to X'$ is given by $x \mapsto (\pi(x), \mu(x))$. Since μ is an embedding, ψ is an embedding as well.

We have shown that $\hat{X}$ is the monoidal transform of X with respect to Σ, and that $\Sigma \circ \pi = \mathbf{E}$. Furthermore, the metric induced on E^* by the inclusion $0 \to E^* \to \hat{X} \times \mathbf{C}^q$ agrees with the metric on E^* obtained by pulling back the metric induced on $V(\Sigma)$ by the inclusion $0 \to V(\Sigma) \to X \times \mathbf{C}^q$. By the proposition, $\hat{X} = P(V(\Sigma))$ and $E^* = V(\Sigma \circ \pi) = L(V(\Sigma))$. Thus, the trivial metric on $X \times \mathbf{C}^q$ induces a metric on $V(\Sigma)$ which in turn induces a metric on $L(V(\Sigma))$ with negative definite curvature. $V(\Sigma)$ is therefore negative; the proof is completed.

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