RESOLUTIONS OF H -CLOSED SPACES

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ABSTRACT. It is shown that every Hausdorff space X is the perfect, irreducible, continuous image of a Hausdorff space $\tilde{X}$ which has a basis with open closures. Further, $w(\tilde{X}) < w(X)$, where $w(\tilde{X})$ represents the weight of $\tilde{X}$, and if X is H -closed then $\tilde{X}$ is also H -closed. A corollary of this result is that if $f: X \rightarrow Y$ is a continuous map of the H -closed space X onto the semi-regular Hausdorff space Y , then $w(Y) < w(X)$.

A Hausdorff space is H -closed if it is closed in every Hausdorff space in which it is embedded. A space is *extremally disconnected* if the closure of every open set is open. An extremally disconnected Hausdorff space $\tilde{X}$ will be called a *continuous* or θ -*continuous resolution* of a space X if there is a function $f: \tilde{X} \rightarrow X$ which is continuous or θ -continuous respectively and also perfect, irreducible, and onto. (See [6] for the definition of θ -continuous.) $w(X)$ will denote the weight of a space X .

It is known that: (1) ([9] and [11]) every regular T_1 space X has a continuous resolution which is compact whenever X is; (2) [6] every Hausdorff space has a θ -continuous resolution which is compact whenever X is H -closed; (3) [8] every Hausdorff space is the continuous image of an extremally disconnected Hausdorff space which is H -closed whenever X is. It follows from the result of V. Ponomarev [9, Theorem 1] that for a space X , the weight of the resolution in (1), (2), or (3) above is less than or equal to $2^{w(X)}$. Further, Ponomarev proved that this is the best possible estimate.

The purpose of this note is to show that by a slight weakening of the extremal disconnectedness of the pre-image, it is possible to obtain a much better control on its weight. A corollary of the main result is that if $g: X \rightarrow Y$ is a continuous map of the H -closed space X onto the semiregular Hausdorff space Y , then $w(Y) \leq w(X)$. We note that instead of using spaces of open ultrafilters, as in [5], [6], [8], and [11], or the methods of [9], the proof of the main theorem uses inverse limit techniques similar to those employed by Friedler and Pettey [4] and Dickman and Vinson [2]. Notation and terminology will follow [4].

DEFINITIONS. A function $f: X \rightarrow Y$ is *semi-open* [7] if $\text{int } f(U)$ is nonempty whenever U is a nonempty open subset of X . A function $f: X \rightarrow Y$ is *irreducible* if f is onto and for no proper closed subset F of X is $f(F) = Y$. Note that "irreducible" has a slightly different meaning in [8].

Received by the editors October 5, 1976 and, in revised form, February 6, 1978.

AMS (MOS) subject classifications (1970). Primary 54B25, 54D25.

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LEMMA 1. *Let $f: X \rightarrow Y$ be a closed, irreducible, and onto map. Then f is semi-open.*

PROOF. Let U be a nonempty open subset of X . Then $X - U$ is a proper closed subset, hence $f(X - U) \neq Y$ and $Y - f(X - U)$ is a nonempty open subset of $f(U)$.

LEMMA 2. *Let V be an open subset of the Hausdorff space Y , let X be the disjoint union of $\text{Cl}_Y(Y - \text{Cl}_Y V)$ and $\text{Cl}_Y V$, and let f be the natural map of X onto Y . Then f is perfect, irreducible, semi-open, continuous and onto, and X is H -closed whenever Y is H -closed.*

PROOF. It is easily verified that f is perfect, irreducible, continuous and onto and Lemma 1 implies that f is semi-open. If Y is H -closed, both $\text{Cl}_Y V$ and $\text{Cl}_Y(Y - \text{Cl}_Y V)$ are the closures of open subsets of Y , and hence H -closed [1, Theorem 3.3]. Then X , as the disjoint union of two H -closed spaces, is also H -closed.

In the sequel we may refer to V as a subset of X as well as of Y .

LEMMA 3. *If $f: X \rightarrow Y$ is a semi-open, onto, continuous map and $\text{Cl } V$ is open in Y , then $f^{-1}(\text{Cl } V) = \text{Cl } f^{-1}V$.*

PROOF. The inclusion $\text{Cl } f^{-1}V \subset f^{-1}(\text{Cl } V)$ follows from continuity, so let $x \in f^{-1}(\text{Cl } V)$ and let U be a neighborhood of x . Then $f(x) \in \text{Cl } V$, which is open. By continuity, x has a neighborhood $U' \subset U$ such that $f(U') \subset \text{Cl } V$. Since f is semi-open, $\text{int } f(U') \neq \emptyset$. But if $\text{int } f(U') \cap \text{Cl } V \neq \emptyset$, then $\text{int } f(U') \cap V \neq \emptyset$. If $y \in \text{int } f(U') \cap V$ and $z \in f^{-1}(y) \cap U'$, then $z \in U \cap f^{-1}(V)$ so that $x \in \text{Cl } f^{-1}(V)$.

COROLLARY. *If $f: X \rightarrow Y$ is semi-open, continuous and onto and Y is extremally disconnected, then $f^{-1}(\text{Cl } V) = \text{Cl } f^{-1}(V)$ for every open subset V of Y .*

Maps such that $f^{-1}(\text{Cl } V) = \text{Cl } f^{-1}(V)$ for every open subset V were studied in [4].

The proof of the next lemma is routine.

LEMMA 4. *Let $\{X_\alpha; f_\beta^\alpha\}$ be an inverse system with each bonding map closed, irreducible, and onto and each projection closed and onto. Then each projection is irreducible (and hence semi-open by Lemma 1).*

THEOREM. *Every Hausdorff space X is the continuous, perfect, irreducible, semi-open image of a Hausdorff space $\tilde{X}$ which has a basis with open closures. Further, $w(\tilde{X}) \leq w(X)$, and if X is H -closed then $\tilde{X}$ is H -closed.*

PROOF. Let $\mathfrak{T}$ be the initial ordinal of cardinality $w(X)$ and by [4, Lemma 2.1] let σ be a one-to-one function from $[0, \mathfrak{T}) \times [0, \mathfrak{T})$ onto $[1, \mathfrak{T})$ such that for each pair (α, β) , $\sigma(\alpha, \beta) > \alpha$. We define an inverse system $\{X_\beta; f_\alpha^\beta\}$, $\alpha \leq \beta < \mathfrak{T}$ such that for each $\beta < \mathfrak{T}$, $w(X_\beta) \leq w(X)$. For each $\beta < \mathfrak{T}$ we

shall pick a basis $\{U_{\beta\gamma}: \gamma < \mathfrak{T}\}$ for X_β and shall assume that f_β^β is the identity mapping on X_β .

Let $X_0 = X$ and choose a base $\{U_{0\gamma}: \gamma \in [0, \mathfrak{T})\}$ for X_0 . For each $\beta < \mathfrak{T}$, let Z_β denote the inverse limit of the system $\{X_\alpha; f_\gamma^\alpha, \gamma \leq \alpha \leq \beta\}$, and for each $\alpha < \beta$, let ϕ_α^β denote the projection from Z_β into X_α . If $\sigma^{-1}(\beta) = (\lambda, \eta)$, then $\beta > \lambda$ so $U_{\lambda\eta}$ has already been defined. Let $V_\beta = (\phi_\lambda^\beta)^{-1}(U_{\lambda\eta})$ and let X_β be the disjoint union of $\text{Cl}_{Z_\beta} V$ and $\text{Cl}_{Z_\beta}(Z_\beta - \text{Cl}_{Z_\beta} V_\beta)$. Let p_β be the projection of X_β onto Z_β and for each α less than β , let $f_\alpha^\beta = \phi_\alpha^\beta \circ p_\beta$. Clearly, for $0 \leq \alpha \leq \gamma \leq \beta$, $f_\alpha^\beta = f_\alpha^\gamma f_\gamma^\beta$. From [4, Lemma 2.2] it follows that $w(X_\beta) \leq w(X)$, so choose a base $\{U_{\beta\gamma}: \gamma \in [0, \mathfrak{T})\}$ for X_β .

Let $\tilde{X}$ denote the inverse limit of $\{X_\beta; f_\alpha^\beta, \alpha \leq \beta < \mathfrak{T}\}$. By [4, Lemma 2.2], $w(\tilde{X}) \leq w(X)$.

If, for each $\alpha < \beta$, each bonding map f_γ^α ($\alpha \geq \gamma$) is perfect, irreducible, semi-open, onto, and continuous, then [4, Lemma 2.5] and Lemma 4 imply that Z_β is nonempty and each projection $\phi_\alpha^\beta: Z_\beta \rightarrow X_\alpha$ is perfect, irreducible, semi-open, onto and continuous. Again, by [4, Lemma 2.5] and Lemmas 2 and 4, $\tilde{X}$ is nonempty and each projection $f_\beta: \tilde{X} \rightarrow X_\beta$ is perfect, irreducible, semi-open, onto and continuous. Let $f = f_0$ be the map from $\tilde{X}$ onto X . As a subspace of a product of Hausdorff spaces, $\tilde{X}$ is Hausdorff.

Let U be a neighborhood of $x \in \tilde{X}$. It follows from the construction that $x \in f_\beta^{-1}(V_\beta) \subset U$ for some $\beta < \mathfrak{T}$. Since each projection is semi-open, onto, and continuous and $\text{Cl}_{X_\beta}(V_\beta)$ is open in X_β , Lemma 3 implies that $\text{Cl}_{\tilde{X}} f_\beta^{-1}(V_\beta) = f_\beta^{-1}(\text{Cl}_{X_\beta} V_\beta)$. Hence, $\{f_\beta^{-1}(V_\beta): \beta < \mathfrak{T}\}$ is a basis for $\tilde{X}$ and every member has an open closure.

Finally, if each X_α ($\alpha < \beta$) is H -closed, then since each f_γ^α and each ϕ_α^β is semi-open and onto, it follows from [7, Theorem 3.7] that each Z_β is H -closed and from Lemma 2 that each X_β is H -closed. Again by [7, Theorem 3.7], $\tilde{X}$ is H -closed.

COROLLARY. *Let $g: X \rightarrow Y$ be a continuous map of an H -closed space X onto a semiregular, Hausdorff space Y . Then $w(Y) \leq w(X)$.*

PROOF. Let $\tilde{X}$ be the space constructed in Theorem 1 and let f be the map from $\tilde{X}$ onto X . By Theorem 1, $\tilde{X}$ has a base of cardinality $w(X)$. Consider $F = g \circ f: \tilde{X} \rightarrow Y$. Since $\tilde{X}$ is H -closed, $F^{-1}(y)$ is H -closed for every $y \in Y$ [12, Theorem 1.4]. Let $\{U_\alpha\}$ be a base for $\tilde{X}$ of cardinality $w(X)$ and such that each member has an open closure. If $\{V_\alpha\}$ is the collection of all finite unions of closures of members of $\{U_\alpha\}$, then each V_α , and hence each $\tilde{X} - V_\alpha$, is clopen and so H -closed. Thus, $F(\tilde{X} - V_\alpha)$ is closed in Y for each α . For each α let $W_\alpha = Y - F(\tilde{X} - V_\alpha)$ and let $y \in W$, an open subset of Y . If V is a regularly open neighborhood of y contained in W , then $F^{-1}(y) \subset F^{-1}(V)$, so that by the covering characterization of H -closed spaces, $F^{-1}(y) \subset V_\alpha \subset \text{Cl } F^{-1}(V)$ for some α . It follows that $y \in W_\alpha = \text{int } W_\alpha \subset \text{int } \text{Cl } V = V \subset W$, and hence $\{W_\alpha\}$ is a base for Y of cardinality $w(X)$. But then $w(Y) \leq w(X)$.

REMARKS. It follows from [9, Theorem 1] that the absolute of the closed unit interval is not second countable. (See [6] for the definition of absolute.) This fact, together with [8, Theorem 3.4] and the preceding corollary imply the following.

PROPOSITION. *No second countable, extremally disconnected H -closed space can be mapped onto the closed unit interval by an irreducible, closed continuous mapping.*

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