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# ON THE INSET OF A CONVERGENCE DOMAIN

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ABSTRACT. Let $A$ be a conservative matrix. The main purpose of this article is to give a negative answer to the open question $E$ raised by Macphail and Wilansky [1], namely, "If $\Lambda_A^\perp = I_A$, must $\Lambda_D^\perp = I_D$ for every matrix $D$ with $c_D = c_A$?".

The main purpose of this short article is to give a negative answer to the open question $E$ which was raised by M. S. Macphail and A. Wilansky in [1].

As usual, we use $c$, $c_0$, $l$, respectively, for the set of all convergent sequences, null sequences, absolutely summable sequences. Let $A$ be an infinite matrix. The convergence domain $c_A = \{x\colon Ax \in c\}$ is an $FK$ space. We assume $A$ conservative, that is, $c \subset c_A$. With $a_k$ denoting the $k$th column limit of $A$, we define the inset

$$I_A = \left\{x \in c_A\colon \sum_k a_k x_k \text{ converges}\right\}$$

and

$$\Lambda_A^\perp = \left\{x \in I_A\colon \lim_A x = \sum a_k x_k\right\}.$$

Each $f \in c_A'$ has a representation

$$f(x) = \mu \lim_A x + t(Ax) + sx \qquad \left(t \in l,\, s \in c_A^\beta\right). \tag{1}$$

Let $A$ be the matrix defined by

$$(Ax)_{2n} = 0, \quad (Ax)_{2n-1} = -\frac{2}{n} x_n + \frac{1}{n} x_{n+1}.$$

Plainly $\lim_A x = 0$ ($x \in c_A$) so $\mu$ is not unique, $I_A = c_A$, $\Lambda_A^\perp = I_A$. Define $g(x) = \lim_n 2^{-n} x_n$ on $c_A$. Let $y = (y_n) = Ax$. It is easy to see that

$$\sum_{k=1}^{n} 2^{-(k+1)} k y_{2k-1} = -2^{-1} x_1 + 2^{-(n+1)} x_{n+1}. \tag{2}$$

Put $t = (t_n)$ where $t_{2k} = 0$, $t_{2k-1} = 2^{-(k+1)} k$, $k = 1, 2, 3, \ldots$. Then $t \in l$. From (2) the function $g$ can be expressed as

$$g(x) = t(Ax) + 2^{-1} x_1.$$

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Hence $g \in c'_A$ and so its kernel $g^\perp$ is closed and $g^\perp \supset \bar{c}_0$. On the other hand the sequence $u = (2^n)$ belongs to $c_A$ and $g(u) = 1$, so $u \in c_A \setminus \bar{c}_0$. There exists $f \in c'_A$ with $f(u) \neq 0$ and $f = 0$ on $\bar{c}_0$. By [2, Satz 5.3] there exists a matrix $D$ with $c_D = c_A$ and $f = \lim_D$ since $\mu$ is not unique for $A$. Now $d_k = \lim_D e^k = f(e^k)$, where $e^k$ denotes the sequence $(0, \ldots, 0, 1, 0, \ldots)$ with the 'one' in the $k$th position. Hence $I_D = c_D = c_A$. But

$$\Lambda_D(u) = \lim_D u - \sum d_k u_k = \lim_D u = f(u) \neq 0.$$

Thus $\Lambda_D^\perp \neq I_D$.

## References

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2. K. Zeller, *Allgemeine Eigenschaften von Limitierungsverfahren*, Math. Z. 53 (1951), 463–487.

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