PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 71, Number 2, September 1978

A SELECTION THEOREM FOR MULTIFUNCTIONS

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ABSTRACT. In this paper the following theorem is proved. X is any set, $\mathbf{H}$ is a family of subsets of X which is λ-additive, λ-multiplicative and satisfies the λ-WRP for some cardinal $\lambda > \aleph_0$. Suppose Y is a regular Hausdorff space of topological weight $\leqslant \lambda$ such that given any family of open sets, there is a subfamily of cardinality $< \lambda$ with the same union. Let $F: X \to \mathbf{C}(Y)$, where $\mathbf{C}(Y)$ is the family of nonempty compact subsets of Y, satisfy $\{x: F(x) \cap C \neq \varnothing\} \in \mathbf{H}$ for any closed subset C of Y. Then F admits a $(\mathbf{H} \cap \mathbf{H}^c)_\lambda$-measurable selector.

1. Introduction. Let X be any set, $\mathbf{H}$ a family of subsets of X and τ any cardinal. We say that $\mathbf{H}$ is τ-additive (τ-multiplicative) if whenever $\{A_\alpha: \alpha < \beta\} \subseteq \mathbf{H}$, where $\beta < \tau$, $\bigcup_{\alpha<\beta} A_\alpha (\bigcap_{\alpha<\beta} A_\alpha) \in \mathbf{H}$. $\mathbf{H}^c$ is the family of subsets of X whose complements belong to $\mathbf{H}$ and $\mathbf{H}_\tau$ is the smallest τ-additive family containing $\mathbf{H}$. $\mathbf{H}$ is said to satisfy the τ-weak reduction principle (τ-WRP) if given $\{A_\alpha: \alpha < \beta\} \subseteq \mathbf{H}$, such that $\bigcup_{\alpha<\beta} A_\alpha = X$, where $\beta < \tau$, there exists a pairwise disjoint family of sets $\{B_\alpha: \alpha < \beta\} \subseteq \mathbf{H}$ satisfying $B_\alpha \subseteq A_\alpha$ for all α and $\bigcup_{\alpha<\beta} B_\alpha = X$. $\aleph_0$ and $\aleph_1$ are used to denote the first infinite ordinal and the first uncountable ordinal respectively. (Note that cardinals are considered as initial ordinals.)

If X is any set, $\mathbf{H}$ a family of subsets of X and Y a topological space, then a function f on X into Y is called $\mathbf{H}$-measurable if $f^{-1}(U) \in \mathbf{H}$ for every open subset U of Y. f is called a selector for a multifunction F on X into the family of nonempty subsets of Y if $f(x) \in F(x)$ for all $x \in X$. If $A \subseteq X \times Y$ for any sets X, Y, then A^x denotes the subset of Y given by $\{y: (x,y) \in A\}$. Π_1 denotes the projection to the first coordinate on $X \times Y$.

In this paper we prove the following:

THEOREM. *Let X be any set, $\mathbf{H}$ a family of subsets of X which is λ-additive, λ-multiplicative and satisfies the λ-WRP for some cardinal $\lambda > \aleph_0$. Suppose Y is a regular Hausdorff space of topological weight $\leqslant \lambda$ such that given any family of open sets in Y, there is a subfamily of cardinality $< \lambda$ with the same union. Let $F: X \to \mathbf{C}(Y)$, where $\mathbf{C}(Y)$ is the family of nonempty compact subsets of Y, satisfy $\{x: F(x) \cap C \neq \varnothing\} \in \mathbf{H}$ for any closed subset C of Y. Then F admits a $(\mathbf{H} \cap \mathbf{H}^c)_\lambda$-measurable selector.*

Received by the editors October 12, 1977.

AMS (MOS) subject classifications (1970). Primary 04A05; Secondary 54C60, 54C65.

Key words and phrases. τ-additive family of sets, τ-multiplicative family of sets, τ-weak reduction principle, $\mathbf{H}$-measurable function, selector.

By putting $\lambda = \aleph_1$ we can deduce the following theorem of Sion: Let $\mathbf{H}$ be a family of subsets of a set X and Y a regular T_1 space of topological weight $\leqslant \aleph_1$ such that each family of open subsets of Y admits a countable subfamily with the same union. Let $F: X \to \mathbf{C}(Y)$ be such that $\{x: F(x) \cap C \neq \varnothing\} \in \mathbf{H}$ for every closed set C in Y. Then F admits a $\sigma(\mathbf{H})$-measurable selector where $\sigma(\mathbf{H})$ denotes the smallest σ-algebra containing $\mathbf{H}$.

2. Proof.

LEMMA. *Let X, Y, $\mathbf{H}$ and F be as in the theorem. Let $\{U_\alpha: \alpha$ is a successor ordinal $< \lambda\}$ be an open base for Y such that $U_\alpha \neq \varnothing$ for any α. Then there exists a family $\{A_\alpha: \alpha < \lambda\}$ of subsets of $X \times Y$ satisfying the following*:

(i) *For each α and x, $\varnothing \neq A_\alpha^x \subseteq F(x)$ and A_α^x is compact.*

(ii) *For each α, $\{x: A_\alpha^x \cap C \neq \varnothing\} \in \mathbf{H}$ if $C \subseteq Y$ is closed.*

(iii) *If $\alpha < \beta$, $A_\beta \subseteq A_\alpha$ for all α and β.*

(iv) *If α is a successor ordinal, then there exists $B_\alpha \in \mathbf{H} \cap \mathbf{H}^c$ such that*

$$\left(X \times \bar{U}_\alpha\right) \cap A_\alpha = \left(B_\alpha \times \bar{U}_\alpha\right) \cap A_\alpha = (B_\alpha \times Y) \cap A_\alpha.$$

PROOF. We define the A_α's by induction as follows: $A_0 = \bigcup_x(\{x\} \times F(x))$. Suppose A_β is defined for all $\beta < \alpha$.

Case 1. $\alpha = \beta + 1$ for some β.

For any successor ordinal γ, let $D_\gamma^\beta = \{x: A_\beta^x \cap \bar{U}_\gamma \neq \varnothing\}$. By induction hypothesis, $D_\gamma^\beta \in \mathbf{H}$. Now $X - \bar{U}_{\beta+1} = \bigcup\{\bar{U}_\gamma: \bar{U}_\gamma \subseteq X - \bar{U}_{\beta+1}\} = \bigcup\{U_\gamma: \bar{U}_\gamma \subseteq X - \bar{U}_{\beta+1}\}$. Let $\{U_\gamma: \gamma \in \Gamma_\beta\}$ be a subfamily of $\{U_\gamma: \bar{U}_\gamma \subseteq X - \bar{U}_{\beta+1}\}$ such that cardinality of $\Gamma_\beta < \lambda$ and $\bigcup\{U_\gamma: \gamma \in \Gamma_\beta\} = \bigcup\{U_\gamma: \bar{U}_\gamma \subseteq X - \bar{U}_{\beta+1}\}$. Thus

$$\bigcup_{\gamma \in \Gamma_\beta} U_\gamma = \bigcup_{\gamma \in \Gamma_\beta} \bar{U}_\gamma = X - \bar{U}_{\beta+1}.$$

If $x \notin D_{\beta+1}^\beta$, $A_\beta^x \cap \bar{U}_{\beta+1} = \varnothing$ and hence $A_\beta^x \cap \bar{U}_\gamma \neq \varnothing$ for some $\gamma \in \Gamma_\beta$. Hence $x \in D_\gamma^\beta$ for some $\gamma \in \Gamma_\beta$. Thus $X = D_{\beta+1}^\beta \cup \{D_\gamma^\beta: \gamma \in \Gamma_\beta\}$. By λ-WRP of $\mathbf{H}$, find a pairwise disjoint family of sets $B_{\beta+1}^\beta$, $\{B_\gamma^\beta: \gamma \in \Gamma_\beta\}$ in $\mathbf{H}$ such that $B_{\beta+1}^\beta \subseteq D_{\beta+1}^\beta$, $B_\gamma^\beta \subseteq D_\gamma^\beta$ for $\gamma \in \Gamma_\beta$ and $B_{\beta+1}^\beta \cup \bigcup_{\gamma \in \Gamma_\beta} B_\gamma^\beta = X$. Clearly, $B_{\beta+1}^\beta, B_\gamma^\beta \in \mathbf{H} \cap \mathbf{H}^c$.

Define

$$A_{\beta+1} = \left(\left(B_{\beta+1}^\beta \times \bar{U}_{\beta+1}\right) \cup \bigcup_{\gamma \in \Gamma_\beta} \left(B_\gamma^\beta \times \bar{U}_\gamma\right)\right) \cap A_\beta$$

and $B_{\beta+1} = B_{\beta+1}^\beta$. (i), (iii) and (iv) are clearly satisfied. To check (ii), let $C \subseteq Y$ be closed. Then $\{x: A_{\beta+1}^x \cap C \neq \varnothing\} = \{x: x \in B_{\beta+1}^\beta$ and $A_\beta^x \cap \bar{U}_{\beta+1} \cap C \neq \varnothing\} \cup \bigcup_{\gamma \in \Gamma_\beta}\{x: x \in B_\gamma^\beta$ and $A_\beta^x \cap \bar{U}_\gamma \cap C \neq \varnothing\}$. As $\mathbf{H}$ is λ-multiplicative (and hence $\aleph_0$-multiplicative) and λ-additive, using the induction hypothesis, we see that $\{x: A_{\beta+1}^x \cap C \neq \varnothing\} \in \mathbf{H}$.

Case 2. α is a limit ordinal.

Let $A_\alpha = \bigcap_{\beta<\alpha} A_\beta$. As $\varnothing \neq A_\beta^x \subseteq F(x)$ for $\beta < \alpha$, each A_β^x is compact and $\{A_\beta^x: \beta < \alpha\}$ has the finite intersection property by (iii), it follows that $\varnothing \neq A_\alpha^x \subseteq F(x)$ and A_α^x is compact. Clearly, (iii) is satisfied and (iv) does not need any verification as α is not a successor ordinal.

To check (ii), let $C \subseteq Y$ be closed.

$$\{x: A_\alpha^x \cap C \neq \varnothing\} = \left\{x: \bigcap_{\beta<\alpha} A_\beta^x \cap C \neq \varnothing\right\}$$

$$= \left\{x: \bigcap_{\beta<\alpha} (A_\beta^x \cap C) \neq \varnothing\right\} = \bigcap_{\beta<\alpha} \{x: A_\beta^x \cap C \neq \varnothing\}.$$

The last equality is obtained by using the compactness of $A_\beta^x \cap C$, $\beta < \alpha$. As $\alpha < \lambda$ and $\{x: A_\beta^x \cap C \neq \varnothing\} \in \mathbf{H}$ for $\beta < \alpha$ by induction hypothesis, $\{x: A_\alpha^x \cap C \neq \varnothing\} \in \mathbf{H}$ by λ-multiplicativity of $\mathbf{H}$.

This completes the proof of the lemma.

PROOF OF THE THEOREM. Let U_α, B_α, α is a successor ordinal $< \lambda$ and A_α, $\alpha < \lambda$, be as in the lemma. Put $G = \bigcap_{\alpha<A} A_\alpha$.

Step 1. G is the graph of a function f and f is a selector for F.

By (i) and (iii), $\varnothing \neq G^x \subseteq F(x)$ for all x. We show that for all x, G^x is a singleton. If not, let there exist points (x, y), (x, z) in G where $y \neq z$. Find a basic open set $U_\alpha \subseteq Y$ such that $y \in U_\alpha \subseteq \bar{U}_\alpha \subseteq X - \{z\}$. As $(x, y) \in G \subseteq A_\alpha$, it follows that $(x, y) \in (X \times \bar{U}_\alpha) \cap A_\alpha = (B_\alpha \times \bar{U}_\alpha) \cap A_\alpha$, α being a successor ordinal. Thus $x \in B_\alpha$ and hence $(x, z) \in (B_\alpha \times Y) \cap A_\alpha = (B_\alpha \times \bar{U}_\alpha) \cap A_\alpha$. Therefore $z \in \bar{U}_\alpha$ which is a contradiction. Define $f(x) = y$ if $\{y\} = G^x$.

Step 2. We now have to show that the function $f: X \to Y$ is $(\mathbf{H} \cap \mathbf{H}^c)_\lambda$-measurable.

Let $V \subseteq Y$ be open. $V = \bigcup\{\bar{U}_\alpha: \bar{U}_\alpha \subseteq V\} = \bigcup\{U_\alpha: \bar{U}_\alpha \subseteq V\}$. There exists a subfamily $\{U_\alpha: \alpha \in \Gamma\}$ of $\{U_\alpha: \bar{U}_\alpha \subseteq V\}$ such that cardinality of $\Gamma < \lambda$ and $\bigcup_{\alpha\in\Gamma} U_\alpha = V$. Thus $\bigcup_{\alpha\in\Gamma} \bar{U}_\alpha = V$. Hence

$$f^{-1}(V) = \bigcup_{\alpha\in\Gamma} f^{-1}(\bar{U}_\alpha) = \bigcup_{\alpha\in\Gamma} \left(\Pi_1\left((X \times \bar{U}_\alpha) \cap G\right)\right).$$

It is enough to show that $\Pi_1((X \times \bar{U}_\alpha) \cap G) \in \mathbf{H} \cap \mathbf{H}^c$ for any successor ordinal α.

Fix α.

$$\Pi_1\left((X \times \bar{U}_\alpha) \cap G\right) = \Pi_1\left((X \times \bar{U}_\alpha) \cap \bigcap_{\gamma<\lambda} A_\gamma\right).$$

We first note the $\Pi_1((X \times \bar{U}_\alpha) \cap \bigcap_{\gamma<\lambda} A_\gamma) = \bigcap_{\gamma<\lambda} \Pi_1((X \times \bar{U}_\alpha) \cap A_\gamma)$. Clearly,

$$\Pi_1\left((X \times \bar{U}_\alpha) \cap \bigcap_{\gamma<\lambda} A_\gamma\right) \subseteq \bigcap_{\gamma<\lambda} \Pi_1\left((X \times \bar{U}_\alpha) \cap A_\gamma\right).$$

Let $x \in \bigcap_{\gamma<\lambda} \Pi_1((X \times \bar{U}_\alpha) \cap A_\gamma)$. Then for all $\gamma < \lambda$, $A_\gamma^x \cap \bar{U}_\alpha \neq \varnothing$. As

$A_\gamma^x \cap \bar{U}_\alpha$ is compact for each $\gamma < \lambda$, using (iii), we see that $\bigcap_{\gamma<\lambda}(A_\gamma^x \cap \bar{U}_\alpha) \neq \emptyset$ so that $x \in \Pi_1((X \times \bar{U}_\alpha) \cap \bigcap_{\gamma<\lambda} A_\gamma)$.

Again, using (iii), we obtain

$$\bigcap_{\gamma<\lambda} \Pi_1\big((X \times \bar{U}_\alpha) \cap A_\gamma\big) = \bigcap_{\alpha \leqslant \gamma < \lambda} \Pi_1\big((X \times \bar{U}_\alpha) \cap A_\gamma\big).$$

Hence $\Pi_1((X \times \bar{U}_\alpha) \cap G) = \bigcap_{\alpha \leqslant \gamma<\lambda} \Pi_1((X \times \bar{U}_\alpha) \cap A_\gamma)$. We next prove that

$$\bigcap_{\alpha \leqslant \gamma < \lambda} \Pi_1\big((X \times \bar{U}_\alpha) \cap A_\gamma\big) = \Pi_1\big((X \times \bar{U}_\alpha) \cap A_\alpha\big)$$

Clearly, $\bigcap_{\alpha \leqslant \gamma<\lambda} \Pi_1((X \times \bar{U}_\alpha) \cap A_\gamma) \subseteq \Pi_1((X \times \bar{U}_\alpha) \cap A_\alpha)$. Let

$$x \in \Pi_1\big((X \times \bar{U}_\alpha) \cap A_\alpha\big) = \Pi_1((B_\alpha \times Y) \cap A_\alpha)$$

(by (iv)). Then $x \in B_\alpha$ and hence $A_\alpha^x \subseteq \bar{U}_\alpha$. If $\alpha \leqslant \gamma < \lambda$, $\emptyset \neq A_\gamma^x \subseteq A_\alpha^x \subseteq \bar{U}_\alpha$ so that $A_\gamma^x \cap \bar{U}_\alpha \neq \emptyset$. Hence $x \in \Pi_1((X \times \bar{U}_\alpha) \cap A_\gamma)$ for $\alpha \leqslant \gamma < \lambda$.

Thus

$$\Pi_1\big((X \times \bar{U}_\alpha) \cap G\big) = \Pi_1\big((X \times \bar{U}_\alpha) \cap A_\alpha\big)$$
$$= \Pi_1((B_\alpha \times Y) \cap A_\alpha) = B_\alpha \in \mathbf{H} \cap \mathbf{H}^c.$$

COROLLARY. *If X, Y, $\mathbf{H}$ are as in the theorem and if $F: X \to \mathbf{C}(Y)$ is such that $\{x: F(x) \cap U \neq \emptyset\} \in \mathbf{H}$ for any open $U \subseteq Y$, then F admits a $(\mathbf{H} \cap \mathbf{H}^c)_\lambda$-measurable selector.*

PROOF. Let $\{U_\alpha: \alpha < \lambda\}$ be a base for Y consisting of nonempty open sets and let $C \subseteq Y$ be closed. Then $Y - C = \bigcup\{\bar{U}_\alpha: \bar{U}_\alpha \subseteq Y - C\} = \bigcup\{U_\alpha: \bar{U}_\alpha \subseteq Y - C\}$. Let $\{U_\alpha: \alpha \in \Gamma\}$ be a subfamily of $\{U_\alpha: \bar{U}_\alpha \subseteq Y - C\}$ which has cardinality $< \lambda$ and satisfying $\bigcup_{\alpha\in\Gamma} U_\alpha = Y - C$. Clearly, $\bigcup_{\alpha\in\Gamma} \bar{U}_\alpha = Y - C$.

Now as $F(x)$ is compact, $F(x) \subseteq Y - C$ if and only if there exist $\alpha_1, \ldots, \alpha_n \in \Gamma$ such that $F(x) \subseteq \bigcup_{i=1}^n U_{\alpha_i} \subseteq \bigcup_{i=1}^n \bar{U}_{\alpha_i} \subseteq Y - C$. Thus

$$\{x: F(x) \subseteq Y - C\} = \bigcup_n \bigcup_{(\alpha_1, \ldots, \alpha_n)} \left\{x: F(x) \cap \bigcap_{i=1}^n \bar{U}_{\alpha_i}^c = \emptyset\right\}$$
$$\in ((\mathbf{H}^c)_\lambda)_{\aleph_1} = (\mathbf{H}^c)_\lambda = \mathbf{H}^c$$

as $\mathbf{H}$ is λ-multiplicative. Hence $\{x: F(x) \cap C \neq \emptyset\} \in \mathbf{H}$. Now we can invoke the theorem.

3. Acknowledgement. I am grateful to Dr. Ashok Maitra for his suggestions.

REFERENCES

1. A. Maitra and B. V. Rao, *Selection theorems for partitions of Polish spaces*, Fund. Math. **93** (1976), 47–56.

2. M. Sion, *On uniformization of sets in topological spaces*, Trans. Amer. Math. Soc. **96** (1960), 237–245.

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