POINCARÉ SERIES OF MODULES OVER LOCAL RINGS

GERSON LEVIN

ABSTRACT. There is a well known conjecture, originally made by Kaplansky, that the Poincaré series $P_R^M(z)$ of a finitely generated module M over a local ring R is a rational function. Two reductions of this conjecture are made, one to the case where M is artinian (for a fixed R) and the other to the case where R is artinian.

Introduction. Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. All R -modules considered are assumed to be finitely generated and all unspecified tensor products are over R .

For any R -module M , we may consider the formal power series

$$P_R^M(z) = \sum_{i=0}^{\infty} B_i z^i$$

where $B_i = \dim_k \text{Tor}_i^R(M, k)$. There is a well-known conjecture, originally made by Kaplansky, that $P_R^M(z)$ is a rational function for all local rings R and all R -modules M . We make two reductions of this conjecture.

THEOREM 1. *The series P_R^M is rational for all local rings R and R -modules M if and only if P_R^M is rational for all artinian local rings R and R -modules M .*

THEOREM 2. *For any local ring R , P_R^M is rational for all R -modules M if and only if P_R^M is rational for all artinian R -modules M .*

The proofs will follow after some lemmas. Theorem 1 for the case $M = k$ first appeared in [3]. However, the proof included here is much shorter. In [2], Gulliksen and Ghione note that the author's result together with a result of Gulliksen's imply that P_R^M is rational for all local rings R and R -modules M if and only if P_R^k is rational for all artinian local rings R . Theorem 2 is new.

LEMMA 1. *Let K be the Koszul complex of R and M an R -module. Then there is an integer n_0 such that for all $n \geq n_0$, the induced homomorphism*

$$H(\mathfrak{m}^{n+1}M \otimes K) \rightarrow H(\mathfrak{m}^n M \otimes K)$$

is zero.

PROOF. Since

$$H(\mathfrak{m}^n M \otimes K) = \frac{(\mathfrak{m}^n M \otimes K) \cap Z(M \otimes K)}{\mathfrak{m}^n B(M \otimes K)}$$

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and since $H(\mathfrak{m}^n M \otimes K)$ is a vector space over k ,

$$\mathfrak{m}((\mathfrak{m}^n M \otimes K) \cap Z(M \otimes K)) \subset \mathfrak{m}^n B(M \otimes K).$$

Then by the Artin-Rees lemma, there is an integer n_0 such that for $n \geq n_0$,

$$\begin{aligned} \mathfrak{m}^{n+1}(M \otimes K) \cap Z(M \otimes K) &\subset \mathfrak{m}(\mathfrak{m}^n(M \otimes K) \cap Z(M \otimes K)) \\ &\subset \mathfrak{m}^n B(M \otimes K). \end{aligned}$$

Thus the induced homomorphism

$$H(\mathfrak{m}^{n+1} M \otimes K) \rightarrow H(\mathfrak{m}^n M \otimes K)$$

is zero.

LEMMA 2. *Let A be a differential graded R -algebra and E a differential graded A -module which is free as a graded A -module (i.e. forgetting differentials) and such that $dE \subset \mathfrak{m}E$. Let M and N be R -modules such that $\mathfrak{m}M \subset N \subset M$, and such that the canonical map $g: N \otimes E \rightarrow M \otimes E$ is injective. Then if the induced homomorphism $H(N \otimes A) \rightarrow H(M \otimes A)$ is zero, so is the induced homomorphism $H(N \otimes E) \rightarrow H(M \otimes E)$.*

PROOF. Assume that $H(N \otimes A) \rightarrow H(M \otimes A)$ is zero.

Then

$$f(Z(N \otimes A)) \subset B(M \otimes A), \quad (1)$$

where f is the canonical map $f: N \otimes A \rightarrow M \otimes A$. Define a filtration on E as follows:

$$F_p E = \sum_{i \geq p} A_i E.$$

We first show that

$$g(Z(N \otimes E) \cap (N \otimes F_p E)) \subset B_p(M \otimes A)E + g(N \otimes F_{p+1} E). \quad (2)$$

Let $S_1, S_2, \dots$ be the elements of degree $n - p$ of a homogeneous A -basis for E . Then if $z \in Z(N \otimes E) \cap (N \otimes F_p E)$, there are $a_i \in N \otimes A_p$ such that $z - \sum a_i S_i \in N \otimes F_{p+1} E$. It follows that $\sum (da_i) S_i \in N \otimes F_p E$ so each $da_i = 0$. But then because of (1), each $f(a_i) \in B_p(M \otimes A)$ and since $g(\sum a_i S_i) = \sum f(a_i) S_i$, (2) is proved.

Next, observe that the derivation formula

$$d(xy) = (dx)y + (-1)^{\deg x} x(dy)$$

shows that

$$(d(M \otimes A_{p+1}))E \subset d((M \otimes A_{p+1})E) + (M \otimes A_{p+1})dE$$

and since $dE \subset \mathfrak{m}E$ and $\mathfrak{m}M \subset N$,

$$(d(M \otimes A_{p+1}))E \subset d(M \otimes E) + g(N \otimes F_{p+1} E). \quad (3)$$

Now by (2) and (3) and the injectivity of g ,

$$g(Z(N \otimes E) \cap (N \otimes F_p E)) \\ \subset g(Z(N \otimes E) \cap (N \otimes F_{p+1} E)) + B(M \otimes E).$$

Since E is A -free, $F_0 E = E$ and also for any n , $F_p E \cap E_n = 0$ for $p > n$. Hence $g(Z(N \otimes E)) \subset B(M \otimes E)$ and $H(N \otimes E) \rightarrow H(M \otimes E)$ is zero.

PROOF OF THEOREM 2. By Lemma 1, there exists an integer n_0 such that for $n \geq n_0$

$$H(\mathfrak{m}^{n+1} M \otimes K) \rightarrow H(\mathfrak{m}^n M \otimes K)$$

is zero. Let X be a minimal free resolution of k . By [1], X is free as a graded K -module. Thus Lemma 2 applies showing that

$$H(\mathfrak{m}^{n+1} M \otimes X) \rightarrow H(\mathfrak{m}^n M \otimes X) \quad (4)$$

is zero for $n \geq n_0$. It also follows that

$$H(\mathfrak{m}^{n+1} M \otimes X) \rightarrow H(M \otimes X)$$

is zero for $n \geq n_0$.

Thus the short exact sequences

$$0 \rightarrow \mathfrak{m}^n M \rightarrow M \rightarrow M/\mathfrak{m}^n M \rightarrow 0$$

induce short exact sequences

$$0 \rightarrow \text{Tor}_p^R(M, k) \rightarrow \text{Tor}_p^R(M/\mathfrak{m}^n M, k) \rightarrow \text{Tor}_{p-1}^R(\mathfrak{m}^n M, k) \rightarrow 0$$

for $n \geq n_0 + 1$, showing that

$$P_R^{M/\mathfrak{m}^n M} = P_R^M + z P_R^{\mathfrak{m}^n M}. \quad (5)$$

We now compute $P_R^{\mathfrak{m}^n M}$ for $n \geq n_0 + 1$. Because of (4), the short exact sequences

$$0 \rightarrow \mathfrak{m}^{n+i+1} M \rightarrow \mathfrak{m}^{n+i} M \rightarrow \mathfrak{m}^{n+i} M / \mathfrak{m}^{n+i+1} M \rightarrow 0$$

induce short exact sequences

$$0 \rightarrow \text{Tor}_p^R(\mathfrak{m}^{n+i} M, k) \rightarrow \text{Tor}_p^R(k, k) \otimes (\mathfrak{m}^{n+i} M / \mathfrak{m}^{n+i+1} M) \\ \rightarrow \text{Tor}_{p-1}^R(\mathfrak{m}^{n+i+1} M, k) \rightarrow 0$$

and thus

$$c_{n+i} P_R^k = P_R^{\mathfrak{m}^{n+i} M} + z P_R^{\mathfrak{m}^{n+i+1} M}$$

for all $i \geq 0$ where

$$c_i = \dim_k \mathfrak{m}^i M / \mathfrak{m}^{i+1} M.$$

Let $H(z) = \sum_{i=0}^{\infty} c_{n+i} z^i$. By the Hilbert theory, c_{n+i} is a polynomial in i for large i so $H(z)$ is a rational function. Now

$$P_R^{\mathfrak{m}^n M} = \sum_{i=0}^{\infty} (-1)^i z^i c_{n+i} P_R^k = H(-z) P_R^k. \quad (6)$$

Then (5) and (6) show that for $n \geq n_0 + 1$, P_R^M is rational if both $P_R^{M/\mathfrak{m}^n M}$ and P_R^k are rational. Since $M/\mathfrak{m}^n M$ and k are artinian modules the theorem follows.

PROOF OF THEOREM 1. Assume that all modules over artinian local rings have rational Poincaré series and let R be any local ring and M an R -module. By Theorem 2, it is enough to prove that P_R^M is rational for artinian R -modules M . So assume that $\mathfrak{m}^r M = 0$.

As in the proof of Theorem 1, there is an integer n_0 such that for $n \geq n_0$

$$\mathrm{Tor}^R(\mathfrak{m}^{n+1}, k) \rightarrow \mathrm{Tor}^R(\mathfrak{m}^n, k) \quad (7)$$

is zero. Let $n \geq \max(r, n_0)$.

Then from (6)

$$P_R^{\mathfrak{m}^{n+1}} = H(-z)P_R^k$$

where $H(z) = \sum_{i=0}^{\infty} \dim(\mathfrak{m}^{n+i+1}/\mathfrak{m}^{n+i+2})z^i$. So

$$P_R^{R/\mathfrak{m}^{n+1}} = 1 + zH(-z)P_R^k.$$

We will show that for any R -module N such that $\mathfrak{m}^n N = 0$,

$$P_R^N / \mathfrak{m}^{n+1} = \frac{P_R^N}{1 - z(P_R^{R/\mathfrak{m}^{n+1}} - 1)}. \quad (8)$$

With $N = k$, this shows that P_R^k is rational and thus that $P_R^{R/\mathfrak{m}^{n+1}}$ is rational. Then with $N = M$, it follows that P_R^M is rational.

To prove (8), let X be a minimal algebra resolution [1] of k over R . We put $R^* = R/\mathfrak{m}^{n+1}$, $\mathfrak{m}^* = \mathfrak{m}/\mathfrak{m}^{n+1}$, and $X^* = X \otimes R^*$. For each $i \geq 2$, let F_i be a free R^* -module such that

$$F_i \otimes k \cong H_{i-1}(X^*).$$

Because of (7) $H(\mathfrak{m}^{n+1}X) \rightarrow H(\mathfrak{m}^n X)$ is zero so $H(\mathfrak{m}^n X^*) \rightarrow H(X^*)$ is surjective. Thus we can choose R -homomorphism $\eta: F_i \rightarrow \mathfrak{m}^n X_{i-1}^*$ such that the diagrams

$$\begin{array}{ccc} F_i & \rightarrow & \mathfrak{m}^n X_{i-1}^* \\ \downarrow & & \downarrow \\ F_i \otimes k & \approx & H_{i-1}(X^*) \end{array}$$

commute. Now define a complex $Y = X^* \otimes T(F)$ with differential given by

$$\begin{aligned} d(x \otimes f_1 \otimes \cdots \otimes f_r) &= dx \otimes f_1 \otimes \cdots \otimes f_r \\ &+ (-1)^{\deg x} x \eta(f_1) \otimes f_2 \otimes \cdots \otimes f_r \end{aligned}$$

for $x \in X^*$ and $f_i \in F$.

It is easily seen that $d^2 = 0$ and that

$$d(y \otimes f) = dy \otimes f \pmod{X^*} \quad (9)$$

for $y \in Y$. Clearly $dY \subset \mathfrak{m}^* Y$. The definition of η shows that

$$Z_i(X^*) \subset B_i(X^* \oplus F) \subset B_i(Y)$$

for $i > 0$. If $x + f \in Z_i(X^* \oplus F)$ then $\eta(f) \in B_{i-1}(X^*)$ so

$$f \in \mathfrak{m}^* F_i \subset d(X_1^* \otimes F_i) + X_i^*.$$

Thus

$$Z_i(X^* \oplus F) \subset B_i(X^* \oplus F \oplus (X^* \otimes F)). \quad (10)$$

In particular, $H_1(Y) = H_2(Y) = 0$. Assume that $H_i(Y) = 0$ for $i < p$. By (9)

$$Z_p(Y) \subset X^* \oplus F \oplus \prod_{i=1}^{p-2} (Z_i(Y) \otimes F_{p-i}) \subset X^* \oplus F \oplus \prod_{i=1}^{p-2} d(Y_{i+1} \otimes F_{p-i})$$

by induction. So by (10) Y is acyclic and hence a minimal resolution of k over R^* . Since $\mathfrak{m}^n N = 0$, the differential in $N \otimes Y$ is given by $d(x \otimes f_1 \otimes \cdots \otimes f_n) = dx \otimes f_1 \otimes \cdots \otimes f_n$ for $x \in N \otimes X^*$, $f_i \in F$. Thus

$$H(N \otimes Y) \cong H(N \otimes X^*) \otimes T(F) \cong H(N \otimes X) \otimes T(F).$$

Formula (8) follows.

COROLLARY. *Let R be a local ring and M an R -module of finite length. Then there is an integer n_0 such that for $n \geq n_0$,*

$$P_{R/\mathfrak{m}^n}^M = P_R^M / (1 - z^2 H(-z) P_R^k)$$

where

$$H(z) = \sum_{i=0}^{\infty} \dim_k(\mathfrak{m}^{n+i} / \mathfrak{m}^{n+i+1}) z^i.$$

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DEPARTMENT OF MATHEMATICS, BROOKLYN COLLEGE (CUNY), BROOKLYN, NEW YORK 11210