

THE LARGE CONDITION FOR RINGS WITH KRULL DIMENSION

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ABSTRACT. A module M with Krull dimension is said to satisfy the large condition if for any essential submodule L of M , the Krull dimension of M/L is strictly less than the Krull dimension of M . For a right noetherian ring R with Krull dimension α this is equivalent to the condition that every f.g. uniform submodule of $E(R_R)$ with Krull dimension α is critical. It is also shown that if R is right noetherian with Krull dimension α and if I_0 is a right ideal maximal with respect to $K \dim I_0 < \alpha$, then R satisfies the large condition if and only if I_0 is a finite intersection of cocritical right ideals and I_0 is closed.

1. Introduction and definitions. Let $|M|$ denote the Krull dimension of a module M . A module M with Krull dimension is said to satisfy the *large condition* if for any essential submodule L , $|M/L| < |M|$. According to [6, 6.1] every semiprime ring with Krull dimension satisfies the large condition. In fact, many properties of semiprime rings with Krull dimension stem from the large condition (see [1, 2.6] and [6, §6]) and these properties are, therefore, shared with this class of rings. It is the purpose of this paper to classify rings satisfying the large condition. When possible, results will be stated for modules.

In §2 it is shown that if M is a right noetherian module it suffices to consider the finitely generated uniform submodules of $E(M)$ having Krull dimension $|M|$. If these are critical then M satisfies the large condition. This characterizes right noetherian rings which satisfy the large condition.

According to Goldie [4, 1.5] if R is a finite dimensional ring, then zero is a finite intersection of maximal complement right ideals. When R is a semiprime ring with Krull dimension these ideals are cocritical. Modules having this property, that zero is a finite intersection of cocritical right ideals, are called *semicritical* in [1] and it is shown there that semicritical modules satisfy the large condition. Although not every right noetherian module satisfying the large condition is semicritical, as an example in §3 shows, this concept is useful in the study of modules satisfying the large condition. Thus §3 is devoted to the consideration of semicritical rings and modules. This

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yields an internal characterization of right noetherian rings satisfying the large condition.

All rings in this paper are associative with unit and all modules are unital. Unless stated otherwise module will mean right module. The injective hull of a module M is denoted by $E(M)$ and the singular submodule by $Z(M)$. If M is a module and K a submodule such that M/K is critical, then K is said to be *cocritical*. A certain familiarity with the definitions and basic results concerning Krull dimension as given in [6] is assumed.

2. The large condition. In this section sufficient conditions for a module to satisfy the large condition are obtained. These conditions are shown to characterize right noetherian rings satisfying the large condition.

A module M_R is said to be *semicritical* if there exists a finite set K_1, \dots, K_n of cocritical submodules such that $\bigcap_{i=1}^n K_i = 0$.

2.1 PROPOSITION. *Let R be a ring with Krull dimension and M be a f.g. R -module. If every f.g. uniform submodule U of $E(M)$ with $|U| < |M|$ is critical, then M is semicritical.*

PROOF. Let $U_1 \oplus \dots \oplus U_n$ be an essential direct sum of uniform submodules of M . Let K_i be a complement of U_i containing $U_1 \oplus \dots \oplus U_{i-1} \oplus U_{i+1} \oplus \dots \oplus U_n$. Then K_i is a maximal complement and consequently M/K_i is a f.g. uniform module. Since U_i is isomorphic to an essential submodule of M/K_i , $E(M/K_i) \cong E(U_i) \subset E(M)$. So by hypothesis M/K_i is critical. Clearly $\bigcap_{i=1}^n K_i = 0$.

A submodule N of a module M is said to be *closed* in M if N has no proper essential extensions inside M .

2.2 LEMMA. *Let M be a module with $|M| = \alpha$ and M_0 a closed submodule of M with $|M_0| < \alpha$. If M/M_0 satisfies the large condition, then M satisfies the large condition.*

PROOF. Let L be an essential submodule of M . According to [5, 1.4], M_0 closed implies that $L + M_0/M_0$ is essential in M/M_0 . Therefore $|M/L + M_0| < \alpha = |M/M_0|$. Consequently,

$$\begin{aligned} |M/L| &= \sup\{|M/L + M_0|, |L + M_0/L|\} \\ &= \sup\{|M/L + M_0|, |M_0/L \cap M_0|\} < \alpha = |M|. \end{aligned}$$

2.3 LEMMA. *Let M be a finite dimensional module. If M_0 is a closed submodule of M , then $E(M/M_0) \cong E(M)/E(M_0)$.*

PROOF. Let K be a complement of M_0 . If $M \supset N \supset M_0$ and $N \cap K = 0$, the finite dimensionality of M implies that N is an essential extension of M_0 . Since M_0 is closed, $N = M_0$. Thus $K \cong K + M_0/M_0$ is essential in M/M_0 and $E(K) \cong E(M/M_0)$. Since $E(M) \cong E(K) \oplus E(M_0)$, it follows that $E(K) \cong E(M/M_0) \cong E(M)/E(M_0)$.

2.4 THEOREM. *Let R be a ring with Krull dimension and let M be a right noetherian R -module with $|M| = \alpha$. If every f.g. uniform submodule U of $E(M)$ with $|U| = \alpha$ is critical, then M satisfies the large condition.*

PROOF. Let M_0 be maximal in the collection of submodules of M with $|M_0| < \alpha$. Clearly M_0 is unique and contains every submodule of Krull dimension $< \alpha$. Furthermore, every nonzero submodule of M/M_0 has Krull dimension α .

Let $U_1 \oplus \cdots \oplus U_n$ be an essential direct sum of uniform submodules of M_0 . Then $E(M_0) \cong \sum_{i=1}^n E(U_i)$. For ease of notation identify these injective hulls. Let $x \in E(U_i) \subset E(M_0)$. Then xR is a f.g. uniform submodule of $E(M)$. If $|xR| = \alpha$, then xR is critical. However, $xR \cap M_0 \neq 0$ and $|xR \cap M_0| < \alpha$ which is impossible. Necessarily then $|xR| < \alpha$. So if $h \in E(M_0) = \sum_{i=1}^n E(U_i)$, say $h = \sum_{i=1}^n x_i$ with $x_i \in E(U_i)$, then $|x_i R| < \alpha$ for all i and this implies $|hR| < \alpha$. Thus every f.g. submodule of $E(M_0)$ has Krull dimension $< \alpha$.

It follows from this that M_0 is closed. For if K is an essential extension of M_0 contained in M , then $E(K) = E(M_0)$. If $k \in K$, then the above shows that $|kR| < \alpha$. Since M_0 is maximal among submodules of M having Krull dimension $< \alpha$, $kR \subset M_0$ and consequently $K = M_0$. Thus M_0 is closed in M .

Let $E(M) = H \oplus E(M_0)$. If U is a f.g. uniform submodule of $E(M/M_0)$ with $|U| \leq |M/M_0|$, then U has Krull dimension α . By 2.3 $E(M/M_0) \cong E(M)/E(M_0) \cong H$ and thus U is isomorphic to a f.g. uniform submodule of $H \subset E(M)$. By hypothesis, then U is critical. It follows from 2.1 that M/M_0 is semicritical and by [1, 2.4] then M/M_0 satisfies the large condition. Finally according to 2.2, M satisfies the large condition.

The converse of 2.3 is not true in general. Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then $U = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ is a uniform R -module which contains the simple submodule $S = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Clearly S satisfies the large condition. However, $E(S)$ contains the f.g. uniform submodule U having $|U| = |S| = 0$ and U is clearly not critical. Thus it is possible to have a right noetherian module M which satisfies the large condition but which has f.g. uniform submodules of Krull dimension $|M|$ in its injective hull which are not critical. The converse to 2.3, however, is true for rings.

2.5 COROLLARY. *Let R be a right noetherian ring with $|R| = \alpha$. Then R satisfies the large condition if and only if every f.g. uniform submodule U of $E(R)$ with $|U| = \alpha$ is critical.*

PROOF. Suppose R satisfies the large condition. Let $U = \sum_{i=1}^n x_i R$ be a f.g. uniform submodule of $E(R)$ with $|U| = \alpha$ and let $0 \neq U_0 \subset U$. Then U_0 is essential in U and as a result $I_i = \{r \in R \mid x_i r \in U_0\}$ is essential in R . In as much as $x_i R + U_0/U_0 \cong R/I_i$, the large condition implies that $|x_i R +$

$U_0/U_0| < |R| = \alpha$. Thus $|U/U_0| \leq \sup\{|x_i R + U_0/U_0| : 1 \leq i \leq n\} < \alpha = |U|$. So U is critical.

The converse is a consequence of 2.4.

2.6 COROLLARY. *If R is a right noetherian ring satisfying the large condition then*

(1) *every f.g. submodule of $E(R)$ with Krull dimension $|R|$ satisfies the large condition;*

(2) *every f.g. nonsingular R -module with Krull dimension $|R|$ satisfies the large condition.*

3. Semicritical modules. If M is a semicritical module, then from [1] M satisfies the large condition. However, not every module which satisfies the large condition is semicritical, as can be seen from the following example. Let F be a field, $R_1 = F[x]$ and $R_2 = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then $R = R_1 \times R_2$ is a right noetherian ring which satisfies the large condition. Since $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ is uniform but not critical over R , according to [1, 2.3], R is not semicritical. The aim of this section is to distinguish between semicritical rings and those which satisfy the large condition.

A ring R is called *k-primitive* if R has a finitely generated faithful critical module. Rings which are *k-primitive* of Krull dimension k are considered in [1]. It is shown there that these rings are semicritical and, furthermore, that every semicritical ring is a subdirect product of *k-primitive* rings. This relationship between semicritical rings and *k-primitive* rings can be used to characterize right noetherian, nonsingular semicritical rings.

3.1 THEOREM. *Let R be a nonsingular right noetherian ring. Then R is semicritical if and only if every finitely generated uniform submodule of $E(R)$ is critical.*

PROOF. Suppose R is semicritical. By [1, 3.6] R is a subdirect product of k_i -primitive rings of Krull dimension k_i , say $R/D_1, \dots, R/D_m$. Further these k_i -primitive rings have the property that if C_i is a critical right ideal of R/D_i , then $E(C_i) \cong E(C_j)$ for $i \neq j$. Let $C_{i1} \oplus \dots \oplus C_{in(i)}$ be a maximal direct sum of critical right ideals which miss D_i . Then C_{is} is isomorphic to a critical right ideal of R/D_i for all s . Thus by [2, 2.3] $E(C_{is}) \cong E(C_{it})$ for all $1 \leq s, t \leq n(i)$, but $E(C_{ik}) \not\cong E(C_{jt})$ for $i \neq j$. Consider $C_{jt} \cap D_i$ for $j \neq i$. If $C_{jt} \cap D_i = 0$, then $C_{jt} \cap \sum_{k=1}^{n(i)} C_{ik} \neq 0$ which implies that $E(C_{jt}) \cong E(C_{ik})$ and this is impossible. Thus $C_{jt} \cap D_i \neq 0$. Since $Z(R/D_i)_R = 0$, necessarily $C_{jt} \subset D_i$ for all $j \neq i$ and $1 \leq t \leq n(j)$.

By [7, 4.9], $E(R) \cong \sum_{i=1}^m E(C_{i1}) \oplus \dots \oplus E(C_{in(i)})$. The singular submodule of R is zero and so $Q = E(R)$ is a semisimple artinian ring. As such Q is a direct product of simple artinian rings which are the homogeneous components $E(C_{i1}) \oplus \dots \oplus E(C_{in(i)})$. Thus there exists central idempotents e_1, \dots, e_m with $Q = E(R) = \prod_{i=1}^m e_i Q$ and $e_i Q \cong E(C_{i1}) \oplus \dots \oplus E(C_{in(i)})$.

Consider the mapping $\phi_i: R \rightarrow e_i R$ which is left multiplication by e_i . Clearly

$C_{jk} \subset \text{Ker } \phi_i$ for all $j \neq i, 1 \leq k \leq n(j)$. Since $\sum_{j=1, j \neq i}^m \sum_{k=1}^{n(j)} C_{jk}$ is large in D_i , $Z(e_i R)_R = 0$ implies $D_i \subset \text{Ker } \phi_i$. Finally since $\text{Ker } \phi_i \cap C_{ik} = 0, 1 \leq k \leq n(i)$, D_i is essential in $\text{Ker } \phi_i$. However $Z(R/D_i)_R = 0$ and so by [5, 2.4] D_i is closed which implies $D_i = \text{Ker } \phi_i$. Thus $e_i R \cong R/D_i$ and $e_i R$ is therefore a k_i -primitive ring of Krull dimension k_i .

Let U be a f.g. uniform R -submodule of Q . Then UQ is a uniform Q -submodule of Q and as a result $U \subset UQ \subset e_i Q$ for some i . This implies that e_i acts as an identity on U and consequently U is a f.g. uniform $e_i R$ -module contained in $E(e_i R) = e_i Q$. By [2, 2.4] every finitely generated $e_i R$ -submodule of $e_i Q$ has Krull dimension $|e_i R|$ and by [1, 2.7], $U_{e_i R}$ is critical. Since the module structure of U as an R -module coincides with the module structure of U as an $e_i R$ -module, U is critical as an R -module.

The converse follows from 2.1.

3.2 COROLLARY. *If R is a right noetherian semicritical ring with $Z(R) = 0$, then every f.g. nonsingular R -module is semicritical.*

It is now clear when the class of right noetherian semicritical rings coincides with the class of right noetherian rings satisfying the large condition—when every right ideal of R has the same Krull dimension as R . A ring R with Krull dimension is called *smooth* if for every nonzero right ideal I , $|I| = |R|$.

3.3 PROPOSITION. *Let R be a smooth right noetherian ring. Then R satisfies the large condition if and only if R is semicritical.*

PROOF. Note that if R is a smooth ring satisfying the large condition then $Z(R) = 0$. The remainder follows from 2.4 and 3.1.

Using this relation between semicritical rings and rings satisfying the large condition, it is now possible to obtain an internal characterization of right noetherian rings satisfying the large condition.

3.4 PROPOSITION. *Let R be a right noetherian ring with $|R| = \alpha$ and let I_0 be a maximal right ideal subject to $|I_0| < \alpha$. Then R satisfies the large condition if and only if R/I_0 is semicritical and I_0 is closed.*

PROOF. It is easy to see that I_0 is a two-sided ideal and that R/I_0 is smooth. If $L \supset I_0$ is such that L/I_0 is large in R/I_0 , then L is large in R and so $|R/I_0/L/I_0| = |R/L| < |R| = |R/I_0|$. Thus R/I_0 satisfies the large condition and by 3.3, then R/I_0 is semicritical. Since R/I_0 is smooth and $|R/I_0| = |R|$, $Z(R/I_0)_R = 0$ and so by [5, 2.4], I_0 is closed.

Conversely, if R/I_0 is semicritical, then by [1, 2.4] R/I_0 satisfies the large condition. Since I_0 is closed, R satisfies the large condition by 2.2.

In 3.4 the condition that I_0 be closed is not redundant. Example 2.2 in [3] is a uniform noetherian ring R having Krull dimension 1. It contains a simple two-sided ideal which is maximal among right ideals with Krull dimension $<$

1 and this simply is I_0 . Clearly I_0 is not closed. R/I_0 is critical and, hence, is semicritical. However, R does not satisfy the large condition since I_0 is large in R but $|R/I_0| = |R| = 1$.

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