A CHARACTERIZATION OF C^* -SUBALGEBRAS

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ABSTRACT. Let A be a closed linear subspace of a C^* -algebra B . Adjoin, if necessary, the identity 1 to B . Then A is a C^* -subalgebra if and only if, for each x in A , the elements x^* and $|x| + 1 - ||x| - 1|$ are in A . If 1 is in A , then A is a C^* -subalgebra if and only if $|x|$ is in A for each x in A . Here $|x|$ denotes the unique positive square root of x^*x in B .

This note is entirely devoted to the proof of and a discussion on the following result.

THEOREM. *Let A be a closed linear subspace of a complex C^* -algebra B . A is a C^* -subalgebra of B if and only if, for each x in A , the elements x^* and $|x| + 1 - ||x| - 1|$ are in A .*

Here $|x|$ denotes the unique positive square root of x^*x in B . The element $|x| + 1 - ||x| - 1|$ is to be interpreted as $g(|x|)$, where $g(t) = t + 1 - |t - 1|$, $t \geq 0$. Since $g(0) = 0$, the continuous functional calculus shows that $g(|x|)$ is in B ; see Proposition 1.5.6, p. 11 of Dixmier [3]. In fact it follows that $|x| + 1 - ||x| - 1|$ is in the C^* -algebra generated by x^*x .

PROOF. The necessity is included in the above remark.

SUFFICIENCY. Let x be in A . The equality

$$|x|^2 = \int_0^\infty \left\{ (|x| + t - ||x| - t|) - (|x| + \tfrac{1}{2}t - ||x| - \tfrac{1}{2}t|) \right\} dt$$

proves that $x^*x = |x|^2$ is in A . The polarisation formula

$$4a^*b = |a + b|^2 - |a - b|^2 + i(|a - ib|^2 - |a + ib|^2)$$

shows that a^*b is in A whenever a and b are in A . Since x^* is in A , whenever x is in A , we see that A is a C^* -subalgebra. $\square$

COROLLARY 1. *Let A and B be as in the Theorem. If each element of A is normal, then A is a C^* -subalgebra of B if and only if, for each x in A , the element $|x| + 1 - ||x| - 1|$ is in A .*

PROOF. The sufficiency part needs some explanation. Let x be in A and n in $\mathbb{N}$. Since $x^*x = xx^*$, it follows from the necessity part of the Theorem and from the polar decomposition of x that

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$$\begin{aligned}
\|2x^* - x^*(n|x| + 1 - |n|x| - 1)|\| &= \|(|n|x| - 1| - n|x| + 1)x^*\| \\
&= \|(|n|x| - 1| - n|x| + 1)|x|\| \\
&\leq \sup\{(|nt - 1| - nt + 1)t; 0 \leq t \leq \|x\|\} \\
&\leq (2n)^{-1}.
\end{aligned}$$

So x^* is in A by the proof of the sufficiency part of the Theorem. $\square$

COROLLARY 2. *Let A and B be as in the Theorem. Assume that B has identity 1. If 1 is in A , then A is a C^* -subalgebra if and only if $|x|$ is in A for each x in A .*

PROOF. The necessity is clear and the sufficiency follows from the equality

$$4x^* = |x + 1|^2 - |x - 1|^2 + i(|x - i|^2 - |x + i|^2), \quad x, y \in B. \quad \square$$

COROLLARY 3. *Let A be a closed subspace of a commutative C^* -algebra B . A is a C^* -subalgebra if and only if with each x in A the element $\min(1, |x|)$ is in A too.*

Corollary 3 is an improvement of Proposition in Dellacherie [2, p. 52]. The following result is due to the referee. The result should be compared with R. V. Kadison [4, Theorem 6, p. 499].

COROLLARY 4. *A continuous linear map Λ between two C^* -algebras B and B' is a C^* -algebra homomorphism if and only if*

$$\Lambda(|x| + 1 - |x| - 1) = |\Lambda(x)| + 1 - |\Lambda(x) - 1|$$

for each x in B .

COROLLARY 5. *Let A be a C^* -subalgebra of B . Assume that B has identity 1. Then A contains 1 if and only if, for some x in A , x^{-1} exists in B .*

PROOF. For the sufficiency part we notice that $|x|^{-1}$ exists whenever x^{-1} exists. Hence let x in A be an element which is invertible in B . Then there is $\delta > 0$ such that $|x| \geq \delta$. Hence $2\delta = |x| + \delta - |x| - \delta$ is in A . $\square$

COROLLARY 6. *Let A be a vector space of real-valued functions. Assume that a real-valued function f is in A whenever there exists a sequence $\{f_n; n \in \mathbb{N}\}$ in A such that*

$$\lim_{n \rightarrow \infty} \sup_{x \in \{|f| < m\}} |f(x) - f_n(x)| = 0$$

for each positive m . Then A is an algebra for the pointwise operations if and only if A is a Stone lattice.

Here a Stone lattice A is a vector space of functions for which $\min(1, f)$ is in A for each f in A ; see H. Bauer [1, p. 194].

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